Lecture Notes for Category Theory

Niels van der Weide (vanderweide@inf.elte.hu)

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1.1 Categories and Examples

Definition 1.1. A **category** $\mathcal{C}$ consists of

- a class $\mathcal{C}_0$ of **objects**;
- for all objects $x, y \in \mathcal{C}_0$, a class $\mathcal{C}_1(x, y)$ of **morphisms**;
- for all objects $x \in \mathcal{C}_0$, a morphism $\mathsf{id}_x : \mathcal{C}_1(x, x)$ called the **identity morphism**;
- for all objects $x, y, z \in \mathcal{C}_0$ and morphisms $f : \mathcal{C}_1(x, y)$ and $g : \mathcal{C}_1(y, z)$, a morphism $g \circ_{\mathcal{C}} f : \mathcal{C}_1(x, z)$ called the **composition**.

We require the following equations

- for all objects $x, y \in \mathcal{C}_0$ and morphisms $f : \mathcal{C}_1(x, y)$, we have $\mathsf{id}_y \circ_{\mathcal{C}} f = f$;
- for all objects $x, y \in \mathcal{C}_0$ and morphisms $f : \mathcal{C}_1(x, y)$, we have $f \circ_{\mathcal{C}} \mathsf{id}_x = f$;
- for all objects $w, x, y, z \in \mathcal{C}_0$ and morphisms $f : \mathcal{C}_1(w, x)$, $g : \mathcal{C}_1(x, y)$, and $h : \mathcal{C}_1(y, z)$, we have $h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f) = (h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f$.

The equations $\mathsf{id}_y \circ_{\mathcal{C}} f = f$ and $f \circ_{\mathcal{C}} \mathsf{id}_x = f$ say that the morphism id is a unit element for composition, and we call them the **left and right unitality laws**. The equation $f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) = (f \circ g) \circ_{\mathcal{C}} h$ says that composition is **associative**, and this equation makes $f \circ_{\mathcal{C}} g \circ_{\mathcal{C}} h$ unambiguous.

To lighten the notation, we often write $x \in \mathcal{C}$ instead of $x \in \mathcal{C}_0$. We also write $f : x \to_{\mathcal{C}} y$ instead of $f \in \mathcal{C}_1(x, y)$, and if $\mathcal{C}$ is clear from the context, we write $f : x \to y$ instead. In addition, we often write id instead of id_x if x is clear from the context, and we write $f \cdot g$ for $f \cdot_{\mathcal{C}} g$ if $\mathcal{C}$ is clear from the context.

Digression. In Definition 1.1 we made a distinction between sets and classes. Such a distinction is necessary to define, for instance, the category of sets (Example 1.2). The objects of this category should be all sets, but as there is no sets of all sets, such a category cannot exist if we require the objects of a category to form a set. We solve this problem by relaxing the definition of category to allow the objects and morphisms to form a class rather than a set. Another solution would be restrict this category to all set whose cardinality is at most κ for some cardinal number κ. However, there is a disadvantage to this solution: we need κ to be an inaccessible cardinal in order to show that the resulting category of sets has desirable properties (e.g., Cartesian closed). Throughout these notes, we disregard issues regarding sets versus classes.

Let us discuss some examples of categories. We start by examples of categories that arise from mathematical structures. In Table 1, we summarise the examples of categories that we discuss throughout this section.

Example 1.2. We define the **category of sets**, written as Set, as follows

- the objects of Set are sets;
- the morphisms from a set X to a set Y are functions from X to Y.

The identity morphism $\mathsf{id} : X \to X$ is defined to be the identity function, i.e., $\mathsf{id}(x) = x$. The composition of morphisms $f : X \to Y$ and $g : Y \to Z$ is defined to be the composition of functions, i.e., $(g \circ f)(x) = g(f(x))$. Let $f : X \to Y$ be a morphism. Left unitality ($\mathsf{id} \circ f = f$) follows from the following equations, where $x \in X$ is an arbitrary element.

$$\begin{aligned}(\mathsf{id} \circ f)(x) &= \mathsf{id}(f(x)) \\ &= f(x)\end{aligned}$$

Right unitality ($f \circ \mathsf{id} = f$) is proven similarly.

$$\begin{aligned}(f \circ \mathsf{id})(x) &= f(\mathsf{id}(x)) \\ &= f(x)\end{aligned}$$

Now suppose we have morphisms $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$. To prove associativity, we take an arbitrary $w \in W$, and we use the following equalities.

$$\begin{aligned}(h \circ (g \circ f))(w) &= h((g \circ f)(w)) \\ &= h(g(f(w))) \\ &= (h \circ g)(f(w)) \\ &= ((h \circ g) \circ f)(w)\end{aligned}$$

Hence, Set indeed forms a category.

Example 1.3. A **monoid** consists of a set M together an element $e_M \in M$ and an operation $m : M \times M \to M$ such that the following equations holds

- $m(e_M, x) = x$ for all $x \in M$;
- $m(x, e_M) = x$ for all $x \in M$;
- $m(x, m(y, z)) = m(m(x, y), z)$ for all $x, y, z \in M$.

We denote $m(x, y)$ by $x \cdot y$, and we denote a monoid by M. A **homomorphism** from a monoid M to a monoid N is a function $f : M \to N$ such that $f(e_M) = e_N$ and $f(x \cdot y) = f(x) \cdot f(y)$. The identity function $\mathsf{id} : M \to M$ is a homomorphism, because

$$\mathsf{id}(e_M) = e_M, \qquad \mathsf{id}(x \cdot y) = x \cdot y = \mathsf{id}(x) \cdot \mathsf{id}(y)$$

If $f : M \to N$ and $g : N \to K$ are homomorphisms, then their composition is a homomorphism as well since

$$(g \circ f)(e_M) = g(f(e_M)) = g(e_N) = e_K$$

$$(g \circ f)(x \cdot y) = g(f(x \cdot y)) = g(f(x) \cdot f(y)) = g(f(x)) \cdot g(f(y))$$

Note that two homorphisms $f, g : M \to N$ are the same if their underlying functions are the same, meaning that to prove $f = g$, it suffices to show that $f(m) = g(m)$ for all $m \in M$.

Now we define the **category of monoids**, writen Mon, as follows.

- the objects of Mon are monoids;
- the morphisms from a monoid M to a monoid N are homomorphisms from M to N.

The identity $\mathsf{id} : M \to M$ is given by the identity homomorphism, and the composition is given by the composition of homomorphisms. We prove that Mon is a category in the same way as we prove that Set is a category. To prove left unitality ($\mathsf{id} \circ f = f$ for all homomorphisms f), we take an arbitrary $m \in M$, and we note

$$\begin{aligned}(\mathsf{id} \circ f)(m) &= \mathsf{id}(f(m)) \\ &= f(m)\end{aligned}$$

Example 1.4. A **group** G consists of a set G together an element $e_G \in G$ and operations $i : G \to G$ and $m : G \times G \to G$ such that the following equations holds

- $m(e_G, g) = g$ for all $g \in G$;
- $m(g, e_G) = g$ for all $g \in G$;
- $m(g, m(h, k)) = m(m(g, h), k)$ for all $g, h, k \in G$;
- $m(g, i(g)) = e_G$ for all $g \in G$;
- $m(i(g), g) = e_G$ for all $g \in G$.

We denote $m(g, h)$ by $g \cdot h$, and we denote $i(g)$ by g^{-1}. A **homomorphism** from a group G to a group H is a function $f : G \to H$ such that $f(e_G) = e_H$ and $f(g \cdot h) = f(g) \cdot f(h)$. Note that we can prove $f(g^{-1}) = (f(g))^{-1}$.

We define the **category of groups**, written Group, as follows

- the objects of Group are groups;
- the morphisms from a group G to a monoid H are homomorphisms from G to H.

We prove that Group is a category the same way as we proved that Mon is a category.

Another example of a category is the category PreOrd (Exercise 1.4). Next we look at constructions of categories.

Example 1.5. We write $\mathbb{1}$ for the set $\{*\}$. We define the **unit category**, denoted by unit, as follows.

- The objects of unit are elements of $\mathbb{1}$;
- The morphisms from $*$ to $*$ are elements of $\mathbb{1}$

The identity morphism is given by $*$, and the composition of morphisms is the constant maps that maps everything to $*$. The necessary laws follow directly, because all elements of $\mathbb{1}$ are equal, and hence all morphisms in unit are equal.

Example 1.6. Let $\mathcal{C}$ be a category. We define **opposite category**, denoted by $\mathcal{C}^{\mathsf{op}}$, as follows.

- The objects of $\mathcal{C}^{\mathsf{op}}$ are objects of $\mathcal{C}$;
- The morphisms from x to y in $\mathcal{C}^{\mathsf{op}}$ are morphisms from y to x in $\mathcal{C}$.

Hence, $\mathcal{C}^{\mathsf{op}}$ is $\mathcal{C}$ but with the morphisms reversed. The identity morphism $\mathsf{id}_x : \mathcal{C}_1^{\mathsf{op}}(x,x)$ in $\mathcal{C}^{\mathsf{op}}$ is the identity morphism $\mathsf{id}_x : \mathcal{C}_1(x,x)$ in $\mathcal{C}$. Suppose that we have morphisms $f : \mathcal{C}_1^{\mathsf{op}}(x,y)$ and $g : \mathcal{C}_1^{\mathsf{op}}(y,z)$, which are the same as morphisms $f : \mathcal{C}_1(y,x)$ and $g : \mathcal{C}_1(z,y)$. We define the composition $g \circ_{\mathcal{C}^{\mathsf{op}}} f$ to be $f \circ_{\mathcal{C}} g$. Left unitality is proven as follows

$$\begin{aligned} \mathsf{id} \circ_{\mathcal{C}^{\mathsf{op}}} f &= f \circ_{\mathcal{C}} \mathsf{id} \\ &= f \end{aligned}$$

Here we use right unitality of $\mathcal{C}$. Right unitality for $\mathcal{C}^{\mathsf{op}}$ is left as an exercise. To prove associativity for $\mathcal{C}^{\mathsf{op}}$, we assume that we have morphisms $f : \mathcal{C}_1^{\mathsf{op}}(w,x)$, $g : \mathcal{C}_1^{\mathsf{op}}(x,y)$, and $h : \mathcal{C}_1^{\mathsf{op}}(y,z)$. Note that we have the following equalities.

$$\begin{aligned} h \circ_{\mathcal{C}^{\mathsf{op}}} (g \circ_{\mathcal{C}^{\mathsf{op}}} f) &= (g \circ_{\mathcal{C}^{\mathsf{op}}} f) \circ_{\mathcal{C}} h \\ &= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h \\ &= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) \\ &= (g \circ_{\mathcal{C}} h) \circ_{\mathcal{C}^{\mathsf{op}}} f \\ &= (h \circ_{\mathcal{C}^{\mathsf{op}}} g) \circ_{\mathcal{C}^{\mathsf{op}}} f \end{aligned}$$

The compositions above make sense, because $f : \mathcal{C}_1^{\mathsf{op}}(w,x)$, $g : \mathcal{C}_1^{\mathsf{op}}(x,y)$, and $h : \mathcal{C}_1^{\mathsf{op}}(y,z)$ are the same as morphisms $f : \mathcal{C}_1(x,w)$, $g : \mathcal{C}_1(y,x)$, and $h : \mathcal{C}_1(z,y)$ in $\mathcal{C}$.

Example 1.7. Suppose that we have categories $\mathcal{C}_1$ and $\mathcal{C}_2$. We define the **product category**, denoted by $\mathcal{C}_1 \times \mathcal{C}_2$, as follows.

- The objects of $\mathcal{C}_1 \times \mathcal{C}_2$ are pairs (x,y) where $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$;
- The morphisms from (x_1,y_1) to (x_2,y_2) in $\mathcal{C}_1 \times \mathcal{C}_2$ are pairs (f,g) of morphisms where $f : x_1 \to_{\mathcal{C}_1} y_1$ and $g : x_2 \to_{\mathcal{C}_2} y_2$.

We leave it as an exercise to construct the identity morphisms and the composition of morphisms, and to prove that it indeed forms a category.

Example 1.8. Let $\mathcal{C}$ be a category and let P be a predicate on the objects of $\mathcal{C}$. We define the **full subcategory** $\mathsf{Full}(\mathcal{C}, \mathsf{P})$ as follows.

- The objects of $\mathsf{Full}(\mathcal{C}, \mathsf{P})$ are objects of $x \in \mathcal{C}$ such that $\mathsf{P}(x)$
- Given objects $x, y \in C$ such that $\mathsf{P}(x)$ and $\mathsf{P}(y)$, then morphisms from x to y in $\mathsf{Full}(\mathcal{C}, \mathsf{P})$ are the same as morphisms $x \to_{\mathcal{C}} y$.

The identity morphisms and composition operations in $\mathsf{Full}(\mathcal{C}, \mathsf{P})$ are inherited from $\mathcal{C}$, and we leave it as an exercise to the reader to verify that we indeed have a category.

Next we show that categories generalize certain mathematical structures. Specifically, we show how to construct categories from other mathematical structures.

Example 1.9. A **preorder** consists of a set X together with a relation $\leq$ on X that is reflexive ($x \leq x$ for each x) and transitive (if $x \leq y$ and $y \leq z$, then $x \leq z$). Every preorder $(X, \leq)$ gives rise to a category $\widetilde{(X, \leq)}$ as follows.

- The objects of $\widetilde{(X, \leq)}$ are elements of X;
- There is a unique morphism from $x \in X$ to $y \in X$ if and only if $x \leq y$.

Since $\leq$ is reflexive, we have an identity morphism. To construct the composition of morphisms in $\widetilde{(X, \leq)}$, we use that $\leq$ is transitive. The desired laws follow from the fact that morphisms in $\widetilde{(X, \leq)}$ are unique.

Example 1.10. Let M be a monoid. We define the category $\mathsf{B}(M)$ as follows.

- The objects of $\mathsf{B}(M)$ is $\mathbb{1}$;
- morphisms from $*$ to $*$ are the same as elements of M

The identity morphism is given by e_M, and if we have morphisms $m_1, m_2 : * \to$ *unitt*, then their composition $m_1 \circ_{\mathsf{B}(M)} m_2$ is given by $m_1 \cdot m_2$. The laws are proven as follows.

$$m \circ_{\mathsf{B}(M)} \mathsf{id} = m \cdot e_M = m$$

$$\mathsf{id} \circ_{\mathsf{B}(M)} m = e_M \cdot m = m$$

$$m_1 \circ_{\mathsf{B}(M)} (m_2 \circ_{\mathsf{B}(M)} m_3) = m_1 \cdot (m_2 \cdot m_3) = (m_1 \cdot m_2) \cdot m_3 = (m_1 \circ_{\mathsf{B}(M)} m_2) \circ_{\mathsf{B}(M)} m_3$$

Since every group is in particular a monoid, we can instantiate Example 1.10 to groups as well.

Example 1.11 (Extra)**.** For the final example, we assume familiarity with the **λ-calculus**. Recall that the terms of the λ-calculus are defined using the following grammar

$$\Lambda := x \mid \Lambda\Lambda \mid \lambda x.\Lambda$$

Arbitrary terms are denoted by M. Types in the λ-calculus are defined using the following grammar.

$$\mathsf{Ty} := \iota \mid \mathsf{Ty} \Rightarrow \mathsf{Ty}$$

Here ι is the base type, and we denote arbitrary types by τ and σ. The typing relation $x_1 : \tau_1, \ldots, x_n : \tau_n \vdash M : \sigma$ is defined using the usual derivation rules We define the **syntactic category of the simply typed λ-calculus**, denoted by STLC, as follows.

- The objects of STLC are types
- The morphisms from τ to σ are equivalence classes of terms t such that $x : \tau \vdash t : \sigma$, where terms are identified using $\beta\eta$-conversion.

The identity morphism from τ to τ is given by the variable $x : \tau \vdash x : \tau$, and if we have terms $x : \tau_1 \vdash s : \tau_2$ and $y : \tau_2 \vdash t : \tau_3$, then their composition is $x : \tau_1 \vdash t[y \mapsto s] : \tau_3$.

Digression. Example 1.1 is inspiring: we can turn various typed languages into categories. For instance, we can extend to include various types, such as sum-types and product types. One might wonder: can we do the same for Haskell? Can we make a category whose objects are types (as in Haskell) and whose morphisms are Haskell functions. Doing so comes with various complications. Since Haskell's type system includes polymorphism, we need to take into account that Haskell types could have type variables. However, there is an even more fundamental issue. Haskell has a function $\mathsf{seq} : a \to b \to b$ that evaluates its first argument before returning its second, and we also have a function $\mathsf{undefined} : a \to b$ that is undefined for each argument. Now we can show the following

$$\mathsf{seq}\ \mathsf{undefined}\ () = \mathsf{undefined}$$

$$\mathsf{seq}\ (\mathsf{undefined} \circ \mathsf{id})() = ()$$

Note that $\mathsf{undefined} \circ \mathsf{id}$ evaluates to itself, which is why the second equation holds. However, from these equations we can conclude that $\mathsf{undefined}$ and $\mathsf{undefined} \circ \mathsf{id}$ cannot be equal, and thus the category laws do not hold.

When we talk about categories, we generally use the language of **diagrams**. A diagram in a category $\mathcal{C}$ is given by a graph whose nodes are objects in $\mathcal{C}$ and whose edges from x to y are labelled by a morphism $f : x \to_{\mathcal{C}} y$. For instance, suppose that we have a category $\mathcal{C}$ and let v, w, x, y, z be objects in $\mathcal{C}$. Suppose that we have morphisms

$$f_1 : v \to w, \quad f_2 : w \to x, \quad f_3 : x \to z, \quad f_4 : w \to y, \quad f_5 : v \to y, \quad f_6 : y \to z$$

An example of a diagram is given below

$$\begin{tikzcd}
 & w \arrow[r, "f_2"] \arrow[dd, "f_4"'] & x \arrow[dd, "f_3"] \\
v \arrow[ur, "f_1"] \arrow[dr, "f_5"'] & & \\
 & y \arrow[r, "f_6"'] & z
\end{tikzcd}$$

Each path in this graph gives rise to a composition of morphisms. For instance, we have two paths from v to y, namely $f_4 \circ f_1$ and f_1. We have three paths from v to z, which are given by the following compositions

$$f_3 \circ f_2 \circ f_1, \quad f_6 \circ f_4 \circ f_1, \quad f_6 \circ f_5$$

Since composition is associative, every path describes a unique morphism. A diagram is said to be **commutative** if for all nodes x and y every path from x to y denotes the same morphism.

To get familiar with the language of diagrams, we look at other examples of categories.

Example 1.12. Let $\mathcal{C}$ be a category and let x be an object in $\mathcal{C}$. We define the **slice category** $\mathcal{C}/x$ as follows.

- The objects of $\mathcal{C}/x$ are pairs of an object $y \in Y$ together with a morphism $f : y \to x$.
- Given objects $y_1, y_2 \in \mathcal{C}$ and morphisms $f_1 : y_1 \to x$ and $f_2 : y_2 \to x$, then morphisms from (y_1, f_1) to (y_2, f_2) in $\mathcal{C}/x$ are morphisms $g : y_1 \to y_2$ such that the following diagram commutes.

$$\begin{tikzcd} y_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & y_2 \arrow[dl, "f_2"] \\ & x & \end{tikzcd}$$

  Concretely, this means that $f_1 = f_2 \circ g$.

The identity morphism on (y, f) is given by id_y. Note that id_y indeed gives a morphism in $\mathcal{C}/x$ since the following diagram commutes

$$\begin{tikzcd} y \arrow[rr, "\mathsf{id}_y"] \arrow[dr, "f"'] & & y \arrow[dl, "f"] \\ & x & \end{tikzcd}$$

Concretely, we have to show that $f = f \circ \mathsf{id}_y$, which holds because id is the unit for composition.

Now suppose that we have three objects in $\mathcal{C}/x$, which are given by objects $y_1, y_2, y_3 \in \mathcal{C}$ together with morphisms $f_1 : y_1 \to x$, $f_2 : y_2 \to x$, and $f_3 : y_3 \to x$. We also assume that we have morphisms $g_1 : (y_1, f_1) \to_{\mathcal{C}/x} (y_2, f_2)$ and $g_2 : (y_2, f_2) \to_{\mathcal{C}/x} (y_3, f_3)$. This means that we have morphisms $g_1 : y_1 \to_{\mathcal{C}} y_2$ making the following triangle commute.

$$\begin{tikzcd} y_1 \arrow[rr, "g_1"] \arrow[dr, "f_1"'] & & y_2 \arrow[dl, "f_2"] \\ & x & \end{tikzcd}$$

Similarly, we have $g_2 : y_2 \to_{\mathcal{C}} y_3$ making the following triangle commute.

$$\begin{tikzcd}
y_2 \arrow[rr, "g_2"] \arrow[dr, "f_2"'] & & y_3 \arrow[dl, "f_3"] \\
& x &
\end{tikzcd}$$

Their composition in $\mathcal{C}/x$ is given by $g_1 \cdot g_2$. We need to show that the following diagram commutes.

$$\begin{tikzcd}
y_1 \arrow[r, "g_1"] \arrow[dr, "f_1"'] & y_2 \arrow[r, "g_2"] \arrow[d, "f_2" description] & y_2 \arrow[dl, "f_3"] \\
& x &
\end{tikzcd}$$

This diagram commutes, because both triangles commute.

We can make this reasoning concrete by presenting it in an equational style. We need to show that $f_1 = (f_3 \cdot g_1) \cdot g_2$.

$$\begin{aligned}
f_1 &= g_1 \cdot f_2 && \text{since } g_1 \text{ is a morphism in } \mathcal{C}/x \\
&= g_1 \cdot (g_2 \cdot f_3) && \text{since } g_2 \text{ is a morphism in } \mathcal{C}/x \\
&= (g_1 \cdot g_2) \cdot f_3 && \text{associativity}
\end{aligned}$$

Hence, $g_1 \cdot g_2$ satisfies the desired requirement.

We leave it to the reader to verify the category laws.

Example 1.13. Let $\mathcal{C}$ be a category. We define the **arrow category** $\mathcal{C}^{\to}$ as follows.

- The objects of $\mathcal{C}^{\to}$ are triples of $x, y \in \mathcal{C}$ together with $f : x \to y$. We denote such objects by $f : x \to y$.
- Given morphisms $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$, morphisms $f_1 \to_{\mathcal{C}^{\to}} f_2$ are pairs of morphisms $g : x_1 \to x_2$ and $h : y_1 \to y_2$ making the following square commute.

$$\begin{tikzcd}
x_1 \arrow[r, "g"] \arrow[d, "f_1"'] & x_2 \arrow[d, "f_2"] \\
y_1 \arrow[r, "h"'] & y_2
\end{tikzcd}$$

The identity morphism is given by the following square

$$\begin{tikzcd}
x \arrow[r, "\mathsf{id}_x"] \arrow[d, "f"'] & x \arrow[d, "f"] \\
y \arrow[r, "\mathsf{id}_y"'] & y
\end{tikzcd}$$

To define the composition of morphisms, we assume that we have two commutative squares as indicated below.

$$\begin{tikzcd}
x_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & x_2 \arrow[d, "f_2"] \\
y_1 \arrow[r, "h_1"'] & y_2
\end{tikzcd}
\qquad
\begin{tikzcd}
x_2 \arrow[r, "g_2"] \arrow[d, "f_2"'] & x_3 \arrow[d, "f_3"] \\
y_2 \arrow[r, "h_2"'] & y_3
\end{tikzcd}$$

Their composition is defined as the following square.

$$\begin{tikzcd}
x_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & x_2 \arrow[r, "g_2"] \arrow[d, "f_2" description] & x_3 \arrow[d, "f_3"] \\
y_1 \arrow[r, "h_1"'] & y_2 \arrow[r, "h_2"'] & y_3
\end{tikzcd}$$

We leave it as an exercise to the reader to verify that composition and identity indeed give rise to morphisms and to verify the category laws.

Category	Notation	Objects	Morphisms
Category of sets	Set	Sets	Functions
Category of monoids	Mon	Monoids	Homomorphisms
Category of groups	Group	Groups	Homomorphisms
Unit category	unit	Single element	Single morphism
Opposite category	$\mathcal{C}^{\mathsf{op}}$	Objects in $\mathcal{C}$	Morphisms in $\mathcal{C}$ (reverse direction)
Product category	$\mathcal{C}_1 \times \mathcal{C}_2$	Pairs of objects	Pairs of morphisms
Full subcategory	$\mathsf{Full}(\mathcal{C}, \mathsf{P})$	Objects in $\mathcal{C}$	Morphisms in $\mathcal{C}$
Slice category	$\mathcal{C}/x$	Pairs of $y \in \mathcal{C}$ with $f : y \to x$	Commutative triangles
Arrow category		Triples of $x, y \in \mathcal{C}$ and $f : x \to y$	Commutative squares
Preorder	$\widetilde{(X, \leq)}$	Elements in X	Given by $\leq$
Monoids	$\mathsf{B}(M)$	Single element	Elements of M

Table 1: Examples of categories

1.2 Special Morphisms

Definition 1.14. Let $\mathcal{C}$ be a category, and let $x, y : \mathcal{C}$ be objects. We say that $f : x \to y$ is a **monomorphism** if for all objects $w : \mathcal{C}$ and morphisms $g_1, g_2 : w \to y$ such that $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$.

Definition 1.15. Let $\mathcal{C}$ be a category, and let $x, y : \mathcal{C}$ be objects. We say that $f : x \to y$ is a **epimorphism** if for all objects $z : \mathcal{C}$ and morphisms $g_1, g_2 : y \to z$ such that $g_1 \circ f = g_2 \circ f$, we have $g_1 = g_2$.

Definition 1.16. Let $\mathcal{C}$ be a category, and let $x, y : \mathcal{C}$ be objects. We say that $f : x \to y$ is an **isomorphism** if there is a morphism $g : y \to x$ such that $f \circ g = \mathsf{id}_y$ and $g \circ f = \mathsf{id}_x$. The collection of isomorphisms is denoted by $x \cong y$.

To get a better understanding of these classes of morphisms, we study them for concrete instances of categories. First, we look at Set.

Proposition 1.17. *A function $f : X \to Y$ of sets is injective if and only if it is a monomorphism in the category* Set.

Proof. Suppose that $f : X \to Y$ is injective, and suppose that we have functions $g_1, g_2 : W \to X$ such that $f \circ g_1 = f \circ g_2$. To show that $g_1 = g_2$, we take $w \in W$, and we must show that $g_1(w) = g_2(w)$. Since f is injective, it suffices to show $f(g_1(w)) = f(g_2(w))$, which holds by the assumption that $g_1 \circ f = g_2 \circ f$.

Next we assume that $f : X \to Y$ is a monomorphism, and we show that it is injective. Suppose that we have $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Note that from x_1, x_2 we get morphisms $\overline{x_1}, \overline{x_2} : \mathbb{1} \to X$ (recall that $\mathbb{1} = \{*\}$):

$$\overline{x_1}(*) = x_1, \quad \overline{x_2}(*) = x_2$$

Note that we have that $f \circ \overline{x_1} = f \circ \overline{x_2}$, since

$$\begin{aligned}(f \circ \overline{x_1})(*) &= f(\overline{x_1}(*) \\ &= f(x_1) \\ &= f(x_2) \\ &= f(\overline{x_2}(*) \\ &= (f \circ \overline{x_1})(*)\end{aligned}$$

Since f is a monomorphism, we must thus have $\overline{x_1} = \overline{x_2}$. Hence,

$$x_1 = \overline{x_1}(*) = \overline{x_2}(* = x_2$$

□

Proposition 1.18. *A function $f : X \to Y$ of sets is surjective if and only if it is an epimorphism in the category* Set.

Proof. Suppose that $f : X \to Y$ is surjective, and suppose that we have functions $g_1, g_2 : Y \to Z$ such that $g_1 \circ f = g_2 \circ f$. We must show that $g_1 = g_2$, so we take an arbitrary $y \in Y$, and we must show $g_1(y) = g_2(y)$. Since f is surjective, we can find $x \in X$ such that $f(x) = y$. We can thus conclude that

$$\begin{aligned} g_1(y) &= g_1(f(x)) \\ &= g_2(f(x)) \\ &= g_2(y) \end{aligned}$$

Here we used our assumption that $g_1 \circ f = g_2 \circ f$.

Next we assume that $f : X \to Y$ is an epimorphism, and we need to show that f is surjective. Let $y \in Y$ be arbitrary. We use classical logic to prove that there is $x \in X$ such that $f(x) = y$, so suppose that for each $x \in X$ we have $f(x) \neq y$. Without loss of generality, we assume that $* \notin Y$. We define a maps $g_1, g_2 : Y \to Y \cup \mathbb{1}$ as follows

$$g_1(y') = y'$$

$$g_2(y') = \begin{cases} y' & \text{if } y \neq y' \\ * & \text{if } y = y' \end{cases}$$

Note that $g_1 \neq g_2$, so to acquire a contradiction, it suffices to show $g_1 \circ f = g_2 \circ f$ since f is an epimorphism. Let $x \in X$. Since $f(x) \neq y$ by assumption, we have

$$g_2(f(x)) = f(x) = g_1(f(x))$$

Hence, f is surjective. □

Proposition 1.19. *A function $f : X \to Y$ of sets is bijective if and only if it is an isomorphism in the category* Set.

Proof. Suppose that $f : X \to Y$ is bijective. To define $g : Y \to X$, we note that for each $y \in Y$ there is a unique x such that $f(x) = y$, and we define $g(y)$ to be that x. From the description of g, we can immediately see that $f(g(y)) = y$: $g(y)$ is the unique element with $f(g(y)) = y$. To show that $g(f(x)) = x$, we use that f is injective, so it suffices to show that $f(g(f(x))) = f(x)$. This follows from the fact that we just showed that $f(g(y)) = y$ where we take y to be $f(x)$.

Next we assume that $f : X \to Y$ is an isomorphism, which means that we have a function $g : Y \to X$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$. To show that f is injective, we assume that we have $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Now we note

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2$$

To show that f is surjective, we let $y \in Y$ be arbitrary. An inverse image for y is given by $g(y)$, since $f(g(y)) = y$ by assumption. □

For the category of monoids and of groups, we can use the same strategy as for the category of sets. We leave this as an exercise (Exercises 1.11).

Next we look at the opposite category $\mathcal{C}^{\mathsf{op}}$.

Proposition 1.20. *A morphism $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ is a monomorphism in $\mathcal{C}^{\mathsf{op}}$ if and only if $f : y \to_{\mathcal{C}} x$ is a epimorphism in $\mathcal{C}$.*

Proof. Suppose that $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ is a monomorphism in $\mathcal{C}^{\mathsf{op}}$. Our goal is to show that $f : y \to_{\mathcal{C}} x$ is an epimorphism in $\mathcal{C}$. Suppose that we have $z : \mathcal{C}$ and $g_1, g_2 : x \to_{\mathcal{C}} z$ such that $g_1 \circ_{\mathcal{C}} f = g_2 \circ_{\mathcal{C}} f$. From g_1 and g_2 we obtain morphisms $g_1, g_2 : z \to_{\mathcal{C}^{\mathsf{op}}} x$. Note the following

$$\begin{aligned} f \circ_{\mathcal{C}^{\mathsf{op}}} g_1 &= g_1 \circ_{\mathcal{C}} f \\ &= g_2 \circ_{\mathcal{C}} f \\ &= f \circ_{\mathcal{C}^{\mathsf{op}}} g_2 \end{aligned}$$

Since f is a monomorphism in $\mathcal{C}^{\mathsf{op}}$, we have $g_1 = g_2$, and thus $f : y \to_{\mathcal{C}} x$ is a epimorphism in $\mathcal{C}$.

Now we assume that $f : y \to_{\mathcal{C}} x$ is an epimorphism in $\mathcal{C}$, and we show that $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ is a monomorphism in $\mathcal{C}^{\mathsf{op}}$. We take the same steps: we assume that we have we have $z : \mathcal{C}$ and $g_1, g_2 : z \to_{\mathcal{C}^{\mathsf{op}}} x$ such that $f \circ_{\mathcal{C}^{\mathsf{op}}} g_1 = f \circ_{\mathcal{C}^{\mathsf{op}}} g_2$. We get morphisms $g_1, g_2 : x \to_{\mathcal{C}} z$ from g_1 and g_2, and we have the following equalities.

$$\begin{aligned} g_1 \circ_{\mathcal{C}} f &= f \circ_{\mathcal{C}^{\mathsf{op}}} g_1 \\ &= f \circ_{\mathcal{C}^{\mathsf{op}}} g_2 \\ &= g_2 \circ_{\mathcal{C}} f \end{aligned}$$

Since f is an epimorphism in $\mathcal{C}$, we get $g_1 = g_2$, which finishes the proof that f is a monomorphism in $\mathcal{C}^{\mathsf{op}}$. □

Proposition 1.21. *A morphism $f : y \to_x$ is a epimorphism in $\mathcal{C}^{\mathsf{op}}$ if and only if f is a monomorphism in $\mathcal{C}$.*

Proof. Left as an exercise (Exercise 1.8). □

Proposition 1.22. *A morphism $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ is an isomorphism in $\mathcal{C}^{\mathsf{op}}$ if and only if $f : y \to_{\mathcal{C}} x$ is an isomorphism in $\mathcal{C}$.*

Proof. Suppose that $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ is an isomorphism in $\mathcal{C}^{\mathsf{op}}$. This means that we have $g : y \to_{\mathcal{C}^{\mathsf{op}}} x$ such that $g \circ_{\mathcal{C}^{\mathsf{op}}} f = \mathsf{id}_x$ and $f \circ_{\mathcal{C}^{\mathsf{op}}} g = \mathsf{id}_y$. Note that g gives us a morphism $g : x \to_{\mathcal{C}} y$ such that

$$f \circ_{\mathcal{C}} g = g \circ_{\mathcal{C}^{\mathsf{op}}} f = \mathsf{id}_x$$

$$g \circ_{\mathcal{C}} f = f \circ_{\mathcal{C}^{\mathsf{op}}} g = \mathsf{id}_y$$

Hence, $f : y \to_{\mathcal{C}} x$ is an isomorphism in $\mathcal{C}$.

The converse is proven similarly. If $f : y \to_{\mathcal{C}} x$ is an isomorphism in $\mathcal{C}$, then we have $g : x \to_{\mathcal{C}} y$ such that $f \circ_{\mathcal{C}} g = \mathsf{id}_x$ and $g \circ_{\mathcal{C}} f = \mathsf{id}_y$. Hence, we have morphisms $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ and $g : y \to_{\mathcal{C}^{\mathsf{op}}} x$ such that

$$g \circ_{\mathcal{C}^{\mathsf{op}}} f = f \circ_{\mathcal{C}} g = \mathsf{id}_x$$

$$f \circ_{\mathcal{C}^{\mathsf{op}}} g = g \circ_{\mathcal{C}} f = \mathsf{id}_y$$

Hence, f is an isomorphism in $\mathcal{C}^{\mathsf{op}}$. □

The final example that we look at, is $\widetilde{(X, \leq)}$.

Proposition 1.23. *Every morphism in $\widetilde{(X, \leq)}$ is a monomorphism and an epimorphism.*

Proof. To show that some morphism f is a monomorphism, we assume that we have $g_1, g_2 : w \to y$ such that $f \circ g_1 = f \circ g_2$. We must also have $g_1 = g_2$, because all morphisms in $\widetilde{(X, \leq)}$ are equal. Similarly, we can show that every morphism is an epimorphism. $\square$

Proposition 1.24. *We have an isomorphism between x and y in $\widetilde{(X, \leq)}$ if and only if $x \leq y$ and $y \leq x$.*

Proof. An isomorphism $\widetilde{(X, \leq)}$ in x and y consists of morphisms $f : x \to y$ and $g : y \to x$. By definition, these morphisms give us that $x \leq y$ and $y \leq x$. If we have morphisms $x \to y$ if $x \leq y$ and $y \to x$ if $y \leq x$, then we obtain an isomorphism, since we have $f : x \to y$ (since $x \leq y$) and $g : y \to x$ (since $y \leq x$). Because all morphisms are equal in $\widetilde{(X, \leq)}$, the necessary equalities hold, and thus we have an isomorphism. $\square$

Proposition 1.25. *Every isomorphism is a monomorphism and an epimorphism.*

Proof. Left as an exercise (Exercise 1.14). $\square$

Note that the converse of Proposition 1.25 is false.

1.3 Exercises

Exercise 1.1. Complete Example 1.4 by proving that Group is a category.

Exercise 1.2. Finish the proof that $\mathcal{C}^{\mathsf{op}}$ (Example 1.6) forms a category by proving the remain unitality law.

Exercise 1.3. Show that $\mathcal{C}_1 \times \mathcal{C}_2$ forms a category (Example 1.7) by defining the identity and composition of morphisms, and proving the category laws.

Exercise 1.4. Define the category of preorders PreOrd where a morphism from $(X, \leq)$ to $(Y, \leq)$ is a function $f : X \to Y$ such that $f(x) \leq f(y)$ whenever $x \leq y$.

Exercise 1.5. Let $\mathcal{C}$ be a category such that for all $f, g : x \to y$, we have $f = g$. Show that $\mathcal{C}$ induces a preorder on the objects of $\mathcal{C}$.

Exercise 1.6. The objects of the **coslice category** $x/\mathcal{C}$ are pairs of objects $y : \mathcal{C}$ together with $x \to y$. Define the morphisms of the coslice category and show that it indeed forms a category

Exercise 1.7. We consider the **category of families** Fam. The objects of Fam consist of a set X and for each $x \in X$ a set $Y(x)$. Morphisms from (X_1, Y_1) to (X_2, Y_2) consist of a function $f : X_1 \to X_2$ together with maps $g_x : Y_1(x) \to Y_2(f(x))$ for each $x \in X_1$. Show that Fam indeed forms a category.

Exercise 1.8. Prove Proposition 1.21.

Exercise 1.9. Let M be a monoid. Show that M is a group if and only if every morphism in $\mathsf{B}(M)$ is an isomorphism.

Exercise 1.10. Characterize the isomorphism, monomorphisms, and epimorphisms in the product category $\mathcal{C}_1 \times \mathcal{C}_2$.

Exercise 1.11. Characterize the isomorphism, monomorphisms, and epimorphisms in the category Mon of monoids.

Exercise 1.12. Characterize the isomorphism, monomorphisms, and epimorphisms in the category Group of groups.

Exercise 1.13. Give a bijective homorphism of preorders that is not an isomorphism.

Exercise 1.14. Prove Proposition 1.25.

Exercise 1.15. Give a category $\mathcal{C}$ and a morphism f in $\mathcal{C}$ such that f is a monomorphism and an epimorphism, but not an isomorphism.

2 Constructions in Categories

2.1 Terminal and Initial Objects

Definition 2.1. Let $\mathcal{C}$ be a category and let $x \in \mathcal{C}$ be an object. We say that x **is terminal** if for all objects y there is a unique morphism $!_y : y \to x$.

The condition that for all objects y we have a unique morphism $!_y : y \to x$, is called the **universal property of terminal objects**.

Definition 2.2. Let $\mathcal{C}$ be a category and let $x \in \mathcal{C}$ be an object. We say that x **is initial** if for all objects y there is a unique morphism $!_y : x \to y$.

The condition that for all objects y we have a unique morphism $!_y : y \to x$, is called the **universal property of initial objects**.

Example 2.3. The set $\mathbb{1}$ is a terminal object in the category Set of sets. If we have any set X, then we can construct a function $f : X \to \mathbb{1}$, which maps every x to the unique element $*$ of $\mathbb{1}$. Now suppose that we have two functions $f, g : X \to \mathbb{1}$. Our goal is to show that $f = g$. To do so, we take an arbitrary $x \in X$, and we need to show that $f(x) = g(x)$. This follows from the following chain of equalities

$$f(x) = * = g(x)$$

since all elements in $\mathbb{1}$ are equal to $*$.

The empty set $\emptyset$ is an initial object in Set. Let X be any set. We have a function f from $\emptyset \to X$, namely the empty function. In addition, all functions $f, g : \emptyset \to X$ are equal, because there is no $x \in \emptyset$.

Proposition 2.4. *Given an object $x : \mathcal{C}$, x is terminal in $\mathcal{C}$ if and only if x is initial in $\mathcal{C}^{\mathsf{op}}$.*

Proof. Suppose x is terminal in $\mathcal{C}$. To prove that x is initial in $\mathcal{C}^{\mathsf{op}}$, we take any $y : \mathcal{C}^{\mathsf{op}}$, and we need to prove there is a unique morphism $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$. Note that morphisms $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ are the same as morphisms $f : y \to_{\mathcal{C}} x$. Since x is terminal, there is a unique morphism $f : y \to_{\mathcal{C}} x$. Hence, there also is a unique morphism $f : x \to_{\mathcal{C}^{\mathsf{op}}} y$ and thus f is initial. In the same way, we prove that x is terminal in $\mathcal{C}$ if x is initial in $\mathcal{C}^{\mathsf{op}}$. □

Proposition 2.5. *Given an object $x : \mathcal{C}$, x is initial in $\mathcal{C}$ if and only if x is terminal in $\mathcal{C}^{\mathsf{op}}$.*

Proof. Exercise. □

Remark. Propositions 2.4 and 2.5 capture an important pattern: the notions of initiality and terminality are dual. Terminality corresponds to initiality, but with the morphisms reversed. In category theory, a wide variety of concepts can be dualised, and we will see plenty of examples throughout these notes.

Example 2.6. We construct a terminal and an initial object in Mon. To construct a terminal object in Mon, we note that $\mathbb{1}$ is a monoid. The unit element $e_{\mathbb{1}}$ is $*$ and the multiplication is the function $\cdot_{\mathbb{1}}$ that maps all pairs (x, y) to $*$. Note that every function $f : M \to \mathbb{1}$ is a homomorphism. We have $f(e_M) = * = e_{\mathbb{1}}$ since all elements in $\mathbb{1}$ are equal, and we have

$$f(x \cdot_M y) = * = * \cdot_{\mathbb{1}} * = f(x) \cdot_{\mathbb{1}} f(y)$$

Hence, there is a unique homomorphism from every monoid M to $\mathbb{1}$.

Next we show that $\mathbb{1}$ also is initial in Mon. Let M be any monoid. We define the function $f : \mathbb{1} \to M$ to be $f(*) = e_M$. Note that f is a homomorphism:

$$f(e_{\mathbb{1}}) = f(*) = e_M$$

Since all $x, y : \mathbb{1}$ are equal to $*$, we also have

$$f(x \cdot_{\mathbb{1}} y) = f(* \cdot_{\mathbb{1}} *) = f(*) = e_M$$

$$f(x) \cdot_M f(y) = f(*) \cdot_M f(*) = e_M \cdot_M e_M = e_M$$

Note that f is unique as well. Let g be any other homomorphism from $*$ to M. Then we have

$$g(*) = g(e_{\mathbb{1}}) = e_M = f(*)$$

Hence, $\mathbb{1}$ is both initial and terminal in Mon.

Example 2.7. Let us study initial and terminal objects in the category $\widetilde{(X, \leq)}$ for $(X, \leq)$ be a preorder. A terminal object for $\widetilde{(X, \leq)}$ is an object $x \in X$ such that for each $y \in X$ there is a unique morphism $f : y \to_{\widetilde{(X,\leq)}} x$. Since morphisms in $\widetilde{(X, \leq)}$ are unique and since we have a morphism $f : y \to_{\widetilde{(X,\leq)}} x$ if and only if $y \leq x$, we have that x is a terminal object in $\widetilde{(X, \leq)}$ if and only if for all y we have $y \leq x$. Hence, x is a maximal element of $(X, \leq)$.

An initial object for $\widetilde{(X, \leq)}$ is an object $x \in X$ such that for each $y \in X$ there is a unique morphism $f : x \to_{\widetilde{(X,\leq)}} y$. Hence, x is an initial object in $\widetilde{(X, \leq)}$ if and only if for all y we have $x \leq y$, meaning that x is a minimal element of $(X, \leq)$.

Proposition 2.8. *Let $\mathcal{C}$ be a category and let x and y be terminal objects in $\mathcal{C}$. Then we have a unique isomorphism $f : x \cong y$.*

Proof. Note that we have $f : x \to y$, because y is terminal, and that we have $g : y \to x$, because x is terminal. These morphisms exist due to the "existence" part in the universal property of terminal objects. We have $f \circ g = \mathsf{id}_y$ and $g \circ f = \mathsf{id}_x$ due to the uniqueness part in the universal property of terminal objects. Both $f \circ g$ and id_y are morphisms to the terminal object y, and hence they must be unique. Similarly for $g \circ f = \mathsf{id}_x$. Note that if we have another morphisms $f' : x \to y$, then we have that $f = f'$, because both are morphisms into the terminal object y. $\square$

Proposition 2.9. *Let $\mathcal{C}$ be a category and let x and y be initial objects in $\mathcal{C}$. Then we have a unique isomorphism $f : x \cong y$.*

Proof. Exercise. $\square$

Digression. In Example 1.1 we looked at the simply typed λ-calculus. The objects in the category that we discussed, are types and morphisms are terms. To construct a terminal object in this category, we need to look at a version of the λ-calculus with unit types. Specifically, our λ-calculus needs to come with a type 1 and a term $* : 1$ such that all terms of type 1 are equal. Similarly, we can study initital objects in this category. For those, we need to have a type 0 together with the usual elimination rule for the empty type.

Hence, to obtain a category with a terminal and initial object from the simply typed λ-calculus, we must look at a version that contains the unit type and the empty type.

2.2 Binary Products and Coproducts

Definition 2.10. Suppose that we have a category $\mathcal{C}$ and objects $x, y \in C$. A **binary product for x and y** consists of

- an object $z \in \mathcal{C}$;
- morphisms $\pi_1 : z \to x$ and $\pi_2 : z \to y$;

satisfying the following universal property: for all objects $w \in \mathcal{C}$ and morphisms $f : w \to x$ and $g : w \to y$ there is a unique morphism $\langle f, g \rangle : w \to z$ with

$$\pi_1 \circ \langle f, g \rangle = f$$

$$\pi_2 \circ \langle f, g \rangle = g$$

We call π_1 and π_2 the **projection morphism**, and we call $\langle f, g \rangle$ the **pairing**. We say that $\mathcal{C}$ has **binary products** if all $x, y \in C$ have a binary product. In case $\mathcal{C}$ has binary products, we denote the binary product of x and y by $x \times y$.

To understand Theorem 2.10, we phrase it using the the language of diagrams. We start with z and the projection morphisms.

$$x \xleftarrow{\pi_1} z \xrightarrow{\pi_2} y$$

We draw the universal property as follows.

$$\begin{tikzcd}
 & w \arrow[dl, "f"'] \arrow[d, dashed, "\exists! \langle f,g \rangle"] \arrow[dr, "g"] & \\
x & z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & y
\end{tikzcd}$$

Definition 2.11. Suppose that we have a category $\mathcal{C}$ and objects $x, y \in C$. A **binary coproduct for x and y** consists of

- an object $z \in \mathcal{C}$;
- morphisms $\iota_1 : x \to z$ and $\iota_2 : y \to z$;

satisfying the following universal property: for all objects $w \in \mathcal{C}$ and morphisms $f : x \to w$ and $g : y \to w$ there is a unique morphism $[f, g] : z \to w$ with

$$[f, g] \circ \iota_1 = f$$

$$[f, g] \circ \iota_2 = g$$

We call ι_1 and ι_2 the **inclusion morphism**, and we call $[f, g]$ the **copairing**. We say that $\mathcal{C}$ has **binary coproducts** if all $x, y \in C$ have a binary coproduct. In case $\mathcal{C}$ has binary coproducts, we denote the binary coproduct of x and y by $x + y$.

We can visualise binary coproducts in a similar way. The inclusion morphisms are drawn as follows.

$$\begin{tikzcd} x \arrow[r, "\iota_1"] & z & y \arrow[l, "\iota_2"'] \end{tikzcd}$$

The universal property is seen as follows.

$$\begin{tikzcd} x \arrow[r, "\iota_1"] \arrow[dr, "f"'] & z \arrow[d, dashed, "\exists! [f,g]"] & y \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\ & w & \end{tikzcd}$$

Example 2.12. We show that the category Set of sets has binary products. Let X and Y be sets, and we define the set $X \times Y$ as follows

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

We claim that $X \times Y$ is the binary product of X and Y. To see why, we define the following morphisms. We start with $\pi_1 : X \times Y \to X$:

$$\pi_1(x, y) = x$$

The morphism $\pi_2 : X \times Y \to Y$ is defined similarly:

$$\pi_2(x, y) = y$$

Next we need to prove the universal property. Let W be any set and suppose that we have functions $f : W \to X$ and $g : W \to Y$. We define $\langle f, g\rangle : W \to X \times Y$ as follows:

$$\langle f, g\rangle(w) = (f(w), g(w))$$

Note that $\pi_1 \circ \langle f, g\rangle = f$, because for every $w \in W$ we have

$$(\pi_1 \circ \langle f, g\rangle)(w) = \pi_1(\langle f, g\rangle(w)) = \pi_1(f(w), g(w)) = f(w)$$

Similarly we prove that $\pi_2 \circ \langle f, g\rangle = g$.

$$(\pi_2 \circ \langle f, g\rangle)(w) = \pi_2(\langle f, g\rangle(w)) = \pi_2(f(w), g(w)) = g(w)$$

We also need to show that $\langle f, g\rangle$ is the unique morphism with this property, so we assume that we have some $h : W \to X \times Y$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Let $w \in W$ be arbitrary, and note that

$$\langle f, g\rangle(w) = (f(w), g(w)) = (\pi_1(h(w)), \pi_2(h(w))) = h(w)$$

Hence, we have $\langle f, g\rangle = h$, meaning that $\langle f, g\rangle$ is the unique morphism with this property. This proves that $X \times Y$ is indeed the binary product of X and Y.

Next we show that the category Set has binary coproducts. Given sets X and Y, we define the set $X + Y$ as follows

$$X + Y = \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$$

We show that $X + Y$ is the coproduct of X and Y. We start by defining the morphism $\iota_1 : X \to X + Y$:

$$\iota_1(x) = (0, x)$$

We define $\iota_2 : Y \to X + Y$ similarly:

$$\iota_2(y) = (1, y)$$

Now we need to prove the universal property, so we take an arbitrary set W and functions $f : X \to W$ and $g : Y \to W$. We define $[f, g] : X + Y \to W$ as follows

$$[f, g](z) = \begin{cases} f(x) & \text{if } z = (0, x) \\ g(y) & \text{if } z = (1, y) \end{cases}$$

Note that $[f, g]$ is total, because all elements in $X + Y$ are either of the form $(0, x)$ or $(1, y)$. We can directly prove the following equations

$$[f, g](\iota_1(x)) = [f, g](0, x) = f(x)$$

$$[f, g](\iota_2(y)) = [f, g](1, y) = g(y)$$

To prove the uniqueness requirement, we assume that we have some $h : X+Y \to W$ such that $h \circ \iota_1 = f$ and $h \circ \iota_2 = g$. If we have $z \in X + Y$, then we know that z is either of the form $(0, x)$ or of the form $(1, y)$ where $x \in X$ or $y \in Y$. In the first case, we have

$$h(0, x) = h(\iota_1(x)) = f(x) = [f, g](0, x) = [f, g](\iota_1(x))$$

In the second case, we have

$$h(1, y) = h(\iota_2(y)) = f(y) = [f, g](1, y) = [f, g](\iota_2(y))$$

This concludes the proof of the uniqueness requirement, and hence $X + Y$ is indeed a binary coproduct of X and Y.

Proposition 2.13. *Let $\mathcal{C}$ be a category and suppose that we have objects $x, y, z \in \mathcal{C}$. Then z is a binary product of x and y in $\mathcal{C}$ if and only if z is a binary coproduct of x and y in $\mathcal{C}^{\mathsf{op}}$. Dually, z is a binary coproduct of x and y in $\mathcal{C}$ if and only if z is a binary product of x and y in $\mathcal{C}^{\mathsf{op}}$.*

Proof. Exercise ☐

Example 2.14. We show that the category Mon has binary products. Let M and N be monoids. We first show that $M \times N = \{(m, n) \mid m \in M, n \in N\}$ is a monoid. We define $e_{M \times N}$ to be $(e_M, e_N$, and we define $(m_1, n_1) \cdot_{M \times N} (m_2, n_2)$ to be $(m_1 \cdot m_2, n_1 \cdot n_2)$. To show that $M \times N$ is the binary product of M and N, we first note that π_1 and π_2 are homomorphisms. For π_1, this is so, because

$$\pi_1(e_{M \times N}) = \pi_1(e_M, e_N) = e_M$$

$$\begin{aligned}
\pi_1((m_1, n_1) \cdot_{M \times N} (m_2, n_2)) &= \pi_1(m_1 \cdot m_2, n_1 \cdot n_2) \\
&= m_1 \cdot m_2 \\
&= \pi_1(m_1, n_1) \cdot \pi_1(m_2, n_2)
\end{aligned}$$

We can use the same reasoning for π_2. If $f : W \to M$ and $g : W \to N$ are homomorphisms, then so is $\langle f, g \rangle$. This is because

$$\begin{aligned}
\langle f, g \rangle (e_W) &= (f(e_W), g(e_W)) \\
&= (e_M, e_N) \\
&= e_{M \times N}
\end{aligned}$$

$$\begin{aligned}
\langle f, g \rangle (w_1 \cdot w_2) &= (f(w_1 \cdot w_2), g(w_1 \cdot w_2)) \\
&= (f(w_1) \cdot f(w_2), g(w_1) \cdot g(w_2)) \\
&= (f(w_1), g(w_1)) \cdot (f(w_2), g(w_2)) \\
&= \langle f, g \rangle (w_1) \cdot \langle f, g \rangle (w_2)
\end{aligned}$$

The uniqueness of $\langle f, g \rangle$ is proven in the same way as for Set.

Constructing binary coproducts in the category of monoids is more challenging, and we leave it as an exercise.

Example 2.15. Let $(X, \leq)$ be a preorder. We look at binary products in $\widetilde{(X, \leq)}$. Given $x, y \in X$, a binary product for x and y is an element $z \in X$ such that $x \leq z$ and $y \leq z$. This z also satisfies another property: for every $w \in X$ with $x \leq w$ and $y \leq w$, we must also have $z \leq w$. Hence, z is a least upper bound of x and y.

We can similarly show that binary coproducts of x and y are greatest lower bounds. Given $x, y \in X$, a binary coproduct for x and y is an element $z \in X$ such that $z \leq x$ and $z \leq y$ such that for all $w \in X$ with $w \leq x$ and $w \leq y$, we have $w \leq z$.

Proposition 2.16. *Let $\mathcal{C}$ be a category and let x and y be objects in $\mathcal{C}$. Suppose that $z_1, z_2 : \mathcal{C}$ are binary products of x and y. Then we have an isomorphism $f : z_1 \cong z_2$.*

Proof. Since z_1 is a binary product of x and y, we have morphisms $\pi_1 : z_1 \to x$ and $\pi_2 : z_2 \to y$. Similarly, we have morphisms $\rho_1 : z_2 \to x$ and $\rho_2 : z_2 \to y$. We first construct a morphism $f : z_1 \to z_2$, which we define to be $\langle \rho_1, \rho_2 \rangle$ as displayed below.

$$\begin{tikzcd}
& z_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dashed, "\exists!\langle\rho_1,\rho_2\rangle" description] & \\
x & & y \\
& z_2 \arrow[ul, "\rho_1"] \arrow[ur, "\rho_2"'] &
\end{tikzcd}$$

We define $g : z_2 \to z_1$ to be $\langle \pi_1, \pi_2 \rangle$.

$$\begin{tikzcd}
& z_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
x & & y \\
& z_2 \arrow[ul, "\rho_1"] \arrow[ur, "\rho_2"'] \arrow[uu, dashed, "\exists!\langle\pi_1,\pi_2\rangle" description] &
\end{tikzcd}$$

We want to prove that $f \circ g = \mathsf{id}_{z_2}$. To do so, it suffices to prove that $f \circ g \circ \rho_1 = \mathsf{id}_{z_2} \circ \rho_1$ and that $f \circ g \circ \pi_2 = \mathsf{id}_{z_2} \circ \rho_2$. This follows from the following equalities

$$\begin{aligned}
f \circ g \circ \rho_1 &= f \circ \langle \pi_1, \pi_2 \rangle \circ \rho_1 \\
&= f \circ \pi_1 \\
&= \langle \rho_1, \rho_2 \rangle \circ \pi_1 \\
&= \rho_1 \\
f \circ g \circ \rho_2 &= f \circ \langle \pi_1, \pi_2 \rangle \circ \rho_2 \\
&= f \circ \pi_2 \\
&= \langle \rho_1, \rho_2 \rangle \circ \pi_2 \\
&= \rho_2
\end{aligned}$$

We prove that $g \circ f = \mathsf{id}_{z_1}$ in the same way. □

Proposition 2.17. *Let $\mathcal{C}$ be a category and let x and y be objects in $\mathcal{C}$. Suppose that $z_1, z_2 : \mathcal{C}$ are binary coproducts of x and y. Then we have an isomorphism $f : z_1 \cong z_2$.*

Proof. Exercise. □

We introduce the following notation. Let $\mathcal{C}$ be a category with binary products. Given $f : x_1 \to x_2$ and $g : y_1 \to y_2$, we define $f \times g : x_1 \times y_1 \to x_2 \times y_2$ as follows.

$$\begin{tikzcd}
x_1 \arrow[d, "f"'] & x_1 \times y_1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "f \times g" description] & y_1 \arrow[d, "g"] \\
x_2 & x_2 \times y_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & y_2
\end{tikzcd}$$

Note we have that $\pi_1 \circ (f \times g) = f \circ \pi_1$ and that $\pi_2 \circ (f \times g) = g \circ \pi_2$. Dually, if $\mathcal{C}$ is a category with binary coproducts, and if we have morphisms $f : x_1 \to x_2$ and $g : y_1 \to y_2$, we define $f + g : x_1 + y_1 \to x_2 + y_2$ as follows.

$$\begin{tikzcd}
x_1 \arrow[r, "\iota_1"] \arrow[d, "f"'] & x_1 + y_1 \arrow[d, dashed, "f+g"] & y_1 \arrow[l, "\iota_2"'] \arrow[d, "g"] \\
x_2 \arrow[r, "\iota_1"'] & x_2 + y_2 & y_2 \arrow[l, "\iota_2"]
\end{tikzcd}$$

We have

$$(f + g) \circ \iota_1 = \iota_1 \circ f$$

$$(f + g) \circ \iota_2 = \iota_2 \circ g$$

Digression. To guarantee that the category defined in Example 1.1 has binary products and binary coproducts, we again need to consider suitable extensions to the type systems. If we want this category to have binary products, we must have a type constructor that all types τ and σ and gives us a type $\tau \times \sigma$. For the type $\tau \times \sigma$, we need to have a first and second projection, but also a pairing operation that allows us to construct terms of type $\tau \times \sigma$. In addition, these terms need to satisfy the relevant β- and η-rules: η rules guarantee uniqueness, whereas β-rules give that the required diagrams commute. For binary coproducts, we need to extend the type system with coproduct types. The extension is dual: we must have types $\tau + \sigma$, left and right inclusion for terms, and a term that allows us to do case distinction. We need to require β- and η-rules to guarantee uniqueness and that the required diagrams commute.

2.3 Pullbacks

Definition 2.18. Let $\mathcal{C}$ be a category with objects $x, y, z \in C$. Suppose we have morphisms $f : x \to z$ and $g : y \to z$. A **pullback** for f and g consists of

- an object $w \in \mathcal{C}$
- morphisms $\pi_1 : w \to x$ and $\pi_2 : w \to y$

such that $f \circ \pi_1 = g \circ \pi_2$. We require the following universal property: for all objects $a \in \mathcal{C}$ with morphisms $h_1 : a \to x$ and $h_2 : a \to y$ such that $h_1 \cdot f = h_2 \cdot g$, there is a unique $k : a \to w$ such that $k \cdot \pi_1 = f$ and $k \cdot \pi_2 = g$.

We say that $\mathcal{C}$ has pullbacks if every f and g has a pullback.

To understand the definition of pullbacks, we visualise the definition using diagrams. We start with the following morphisms.

$$\begin{tikzcd}
 & y \arrow[d, "g"] \\
x \arrow[r, "f"'] & z
\end{tikzcd}$$

Our goal is to say that w is a pullback of f and g, which means that we have a commuting diagram as follows.

$$\begin{tikzcd}
w \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & y \arrow[d, "g"] \\
x \arrow[r, "f"'] & z
\end{tikzcd}$$

The symbol $\lrcorner$ indicates that this square is a pullback, which means that the following universal property is satisfied.

$$\begin{tikzcd}
a \arrow[dr, dashed, "\exists! k"] \arrow[drr, bend left, "h_2"] \arrow[ddr, bend right, "h_1"'] & & \\
& w \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & y \arrow[d, "g"] \\
& x \arrow[r, "f"'] & z
\end{tikzcd}$$

Example 2.19. Let X, Y, Z be sets and suppose that we have functions $f : X \to Z$ and $g : Y \to Z$. The pullback for f and g is the following set

$$W = \{(x, y) \mid x \in X, y \in Y, f(x) = g(y)\}$$

To show that W is indeed the pullback of f and g, we define the following functions. We start with $\pi_1 : W \to X$.

$$\pi_1(x, y) = x$$

Next we define $\pi_2 : W \to Y$.

$$\pi_2(x, y) = y$$

Note that $f \circ \pi_1 = g \circ \pi_2$. Every element of W is of the form (x, y) with $x \in X$, $y \in Y$ and $f(x) = g(y)$. Hence,

$$(f \circ \pi_1)(x, y) = f(\pi_1(x, y)) = f(x) = g(y) = g(\pi_2(x, y)) = (g \circ \pi_2)(x, y)$$

Now suppose that we have some set A with functions $h_1 : A \to X$ and $h_2 : A \to Y$ such that $h_1 \circ \pi_1 = f$ and $h_2 \circ \pi_2 = g$. We define a map $k : A \to W$ as follows

$$k(a) = (h_1(a), h_2(a))$$

Note that $k(a)$ is indeed an element of W since $f(h_1(a)) = g(h_2(a))$ by assumption. In addition,

$$(\pi_1 \circ k)(a) = \pi_1(k(a)) = \pi_1(h_1(a), h_2(a)) = h_1(a)$$

$$(\pi_2 \circ k)(a) = \pi_2(k(a)) = \pi_2(h_1(a), h_2(a)) = h_2(a)$$

To prove that k is unique, we assume that we have $k' : A \to W$ such that $\pi_1 \circ k' = h_1$ and $\pi_2 \circ k' = h_2$. Then we must also have for every $a \in A$

$$k(a) = (h_1(a), h_2(a)) = (\pi_1(k'(a)), \pi_2(k'(a))) = k'(a)$$

Hence, W is indeed a pullback.

Example 2.20. Let $(X, \leq)$ be a preorder. Pullbacks in $\widetilde{(X, \leq)}$ are the same as least upperbounds. If we have $x \leq z$ and $y \leq z$, then a pullback consists of $w \in X$ such that for all $a \in X$ with $a \leq x$ and $a \leq y$, we must also have $a \leq w$. Hence, w is the least upper bound of x and y.

Proposition 2.21. *Pullbacks are unique up to isomorphism.*

Proof. Exercise. □

2.4 Exercises

Exercise 2.1. Prove Proposition 2.5.

Exercise 2.2. Prove Proposition 2.9.

Exercise 2.3. Prove Proposition 2.13.

Exercise 2.4. Let $\mathcal{C}$ be a category with binary products. Construct an isomorphism between $x \times y$ and $y \times x$.

Exercise 2.5. Let $\mathcal{C}$ be a category with binary coproducts. Construct an isomorphism between $x + (y + z)$ and $(x + y) + z$.

Exercise 2.6. Let $\mathcal{C}$ be a category with binary products and a terminal object $\mathbb{1}$. Construct an isomorphism between $x \times \mathbb{1}$ and x.

Exercise 2.7. Let $\mathcal{C}$ be a category with binary coproducts and an initial object $\mathbb{0}$. Construct an isomorphism between $x + \mathbb{0}$ and x.

Exercise 2.8. Let $\mathcal{C}$ be a category with binary products and a terminal object. Show that the following square is a pullback.

$$\begin{tikzcd}
x \times y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & y \arrow[d, "!_y"] \\
x \arrow[r, "!_y"'] & \mathbb{1}
\end{tikzcd}$$

Exercise 2.9. Let $\mathcal{C}$ be a category with binary products. Prove that the following square is a pullback.

$$\begin{tikzcd}
x_1 \times y \arrow[r, "f \times \mathsf{id}_y"] \arrow[d, "\pi_1"'] & x_2 \times y \arrow[d, "\pi_1"] \\
x_1 \arrow[r, "f"'] & x_2
\end{tikzcd}$$

Exercise 2.10. Prove Proposition 2.21

Exercise 2.11. Give the dual notion of pullbacks, called **pushout**.

3 Functors and Natural Transformations

3.1 Functors

Definition 3.1. functor

Example 3.2. identity functor

Example 3.3. composition of functors

Example 3.4. constant functor

Example 3.5. maybe $F(X) = X + 1$

Example 3.6. product $F(X) = X \times A$

Example 3.7. more complicated: $F(X) = X \times A + 1$

Example 3.8. product of functors

Example 3.9. coproduct of functors

Example 3.10. representable functor

Example 3.11. homset functor

Example 3.12. power set

Example 3.13. pullback

Digression. λ-calculus: terms as a presheaf

3.2 Natural Transformations

Definition 3.14. Natural transformation

Example 3.15. identity

Example 3.16. composition

Example 3.17. left whiskering

Example 3.18. right whiskering

Example 3.19. horizontal composition

Example 3.20. first projection of product

Example 3.21. second projection of product

Example 3.22. pairing of natural transformations

Example 3.23. unit for maybe monad

Example 3.24. multiplication for maybe monad

Example 3.25. unit for list monad

Example 3.26. multiplication for list monad

Example 3.27. constant natural transformation

Example 3.28. representable natural transformation

3.3 Categories of Functors

Definition 3.29. the functor category

Proposition 3.30. *This is indeed a category*

Proof. □

Definition 3.31. the category of categories

Proposition 3.32. *This is indeed a category*

Proof. □

Digression. the 2-category of categories

Example 3.33. comma category

3.4 Exercises

Exercise 3.1. opposite functor

Exercise 3.2. coproduct of functors

Exercise 3.3. functor on full subcategory

Exercise 3.4. functor on slice category

Exercise 3.5. functor on arrow category

Exercise 3.6. C to arrow cat is functorial

Exercise 3.7. finite power set
also do unit and counit

Exercise 3.8. distribution monad
also do unit and counit

Exercise 3.9. arrow cat, natural transformations and functors

Exercise 3.10. initial object in the category of categories

Exercise 3.11. terminal object in the category of categories

Exercise 3.12. binary products in the category of categories

Exercise 3.13. binary coproducts in the category of categories

4 Algebras and Coalgebras

5 Adjunctions

6 Monads

7 Presheaves and the Yoneda Lemma

8 Cartesian Closed Categories

9 Cartesian Closed Categories and the λ-Calculus

10 Hyperdoctrines

11 Quantifiers as Adjoints

12 Triposes and Higher-Order Logic