

Category Theory: Exercise Sheet 1

This exercise sheet contains five exercises for which you can earn 100 points in total. You can submit your solutions via email (vanderweide@inf.elte.hu) or by handing them in to me in person. The deadline is **16 October, 23:59**.

Exercise 1 (15pt). Provide a counter example to the following statement: given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a morphism $f : x \rightarrow_{\mathcal{C}} y$, we have that f is an isomorphism whenever $F(f)$ is an isomorphism.

Exercise 2 (15pt). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Construct a functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$.

Exercise 3 (25pt). Let $\mathcal{C}$ be a category with a terminal object and binary products, and let $F : \mathcal{C} \rightarrow \mathcal{C}$ be a functor.

1. Show that the category $\mathbf{Alg}_F$ has a terminal object (**10pt**)
2. Show that the category $\mathbf{Alg}_F$ has binary products (**15pt**)

Exercise 4 (15pt). Recall that the functor $\mathcal{L}_A : \mathbf{Set} \rightarrow \mathbf{Set}$, which is defined to be $\mathcal{L}_A(X) = X \times A + 1$, has an initial algebra $(\mathbf{List}(A), \text{Nil}, \text{Cons})$ for each set A , and we denote this initial algebra by $\mathbf{List}(A)$. Suppose that we have two sets A and B , and a function $f : A \rightarrow B$. Construct a morphism $\bar{f} : \mathbf{List}(A) \rightarrow \mathbf{List}(B)$ using the fact that $\mathbf{List}(A)$ is an initial algebra.

Exercise 5 (30pt). For the final exercise, we look at **rings**. The relevant definitions can be found in the appendix.

1. Show that we have a category **Ring** whose objects are rings and whose morphisms are ring homomorphisms (**5pt**).
2. Show that the category of rings has a terminal object (**5pt**).
3. Show that the category of rings has binary products (**10pt**).
4. Show that the integers with the usual addition and multiplication form an initial object in the category of rings (**10pt**).
Note: you do not need to prove that the integers form a ring.

Appendix with Supplementary Definitions

Definition. A **(commutative) ring** consists of a set R together with elements $0 \in R$ and $1 \in R$, a unary operation $- : R \rightarrow R$ and binary operations $+$: $R \times R \rightarrow R$ and $\cdot$: $R \times R \rightarrow R$ such that

- $x + 0 = x = 0 + x$ and $x \cdot 1 = x = 1 \cdot x$;
- $x + y = y + x$ and $x \cdot y = y \cdot x$;
- $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- $x + (-x) = 0$ and $(-x) + x = 0$
- $x \cdot (y + z) = x \cdot y + x \cdot z$

We write ring instead of commutative ring.

Definition. Given rings R_1 and R_2 , a function $f : R_1 \rightarrow R_2$ is called a **ring homomorphism** if

- $f(0) = 0$
- $f(1) = 1$
- $f(-x) = -f(x)$
- $f(x + y) = f(x) + f(y)$
- $f(x \cdot y) = f(x) \cdot f(y)$