Category Theory: Exercise Sheet 2

This exercise sheet contains four exercises for which you can earn 100 points in total. You can submit your solutions via email (vanderweide@inf.elte.hu) or by handing them in to me in person. The deadline is **20 November, 23:59**.

Exercise 1 (25pt). Let $M_1 : \mathsf{Set} \to \mathsf{Set}$ and $M_2 : \mathsf{Set} \to \mathsf{Set}$ be monads. Show that $N : \mathsf{Set} \to \mathsf{Set}$ with $N(X) = M_1(X) \times M_2(X)$ is a monad.

Exercise 2 (20pt). Let $\mathsf{L} \dashv \mathsf{R}$ be an adjunction where $\mathsf{L} : \mathcal{C} \to \mathcal{D}$. Show that whenever $x \in \mathcal{C}$ is an initial object, then $\mathsf{L}(x)$ is an initial object as well.

Exercise 3 (25pt). Let z be a binary product of x and y. Construct a natural isomorphism between repr_z and the presheaf that maps w to $\mathsf{repr}_x(w) \times \mathsf{repr}_y(w)$.

Exercise 4 (30pt). This exercise is about join semilattices, and the relevant definitions can be found in the appendix. We have a category JSL of join semilattives and homomorphisms, and a functor $U : \mathsf{JSL} \to \mathsf{Set}$ mapping a join semilattice to its carrier set. Show that $U : \mathsf{JSL} \to \mathsf{Set}$ has a left adjoint.

Hint: the left adjoint maps a set X to the set of finite subsets.

Appendix with Supplementary Definitions

Definition. A **join semilattice** consists of a set X together with elements $\bot \in X$ and an operation $\cup : X \times X \to X$ such that

- $x \cup \bot = \bot \cup x = x$ for all $x \in X$;
- $x \cup y = y \cup x$ for all $x, y \in X$;
- $x \cup x = x$ for all $x \in X$;
- $x \cup (y \cup z) = (x \cup y) \cup z$ for all $x, y, z \in X$

Definition. Given join semilattices X_1 and X_2, a function $f : X_1 \to X_2$ is called a **homomorpism** if

- $f(\emptyset) = \emptyset$
- $f(x \cup y) = f(x) \cup f(y)$