Insights From Univalent Foundations
A Case Study Using Double Categories

Nima Rasekh, **Niels van der Weide**, Benedikt Ahrens, Paige Randall North

What does it mean
for mathematical structures
to be the same?

Sameness for Structures

Mathematical structures often are identified up to **isomorphism**

- For instance, we identify groups up to isomorphism
- We can replace isomorphic groups by each other (as if they were equal)
- All properties that we care about in group theory, are **invariant under isomorphism**

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A Case against Set Theoretic Equality

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Disadvantage:

- **we** need to carve out a collection of relevant properties
- and **we** need to prove that these are invariant under isomorphism

Our foundations should do that for us

Univalent Foundations

- **Univalent foundations** (UF) is a flavor of intensional Martin-Löf type theory with the **univalence axiom**[1]
- Univalence axiom: **types are identified if they are equivalent**

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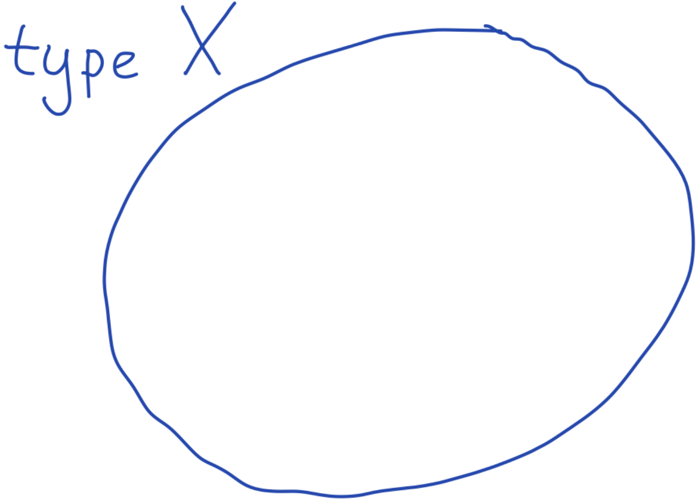
Key features of univalent foundations:

- **Identity is proof relevant**: not every proof of $p = q$ is the same! Proofs of identity carry information
- Interpretation in **homotopy theory**: types as spaces, terms as points, proofs of identity as paths

[1] "Homotopy type theory: Univalent foundations of mathematics"

Types as Spaces



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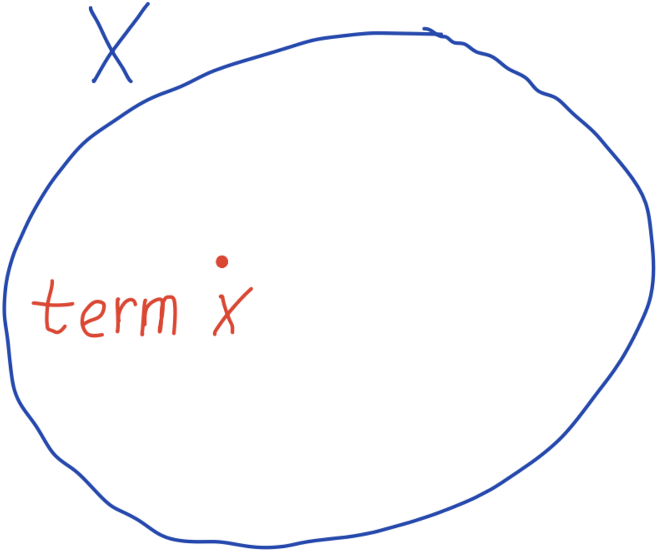

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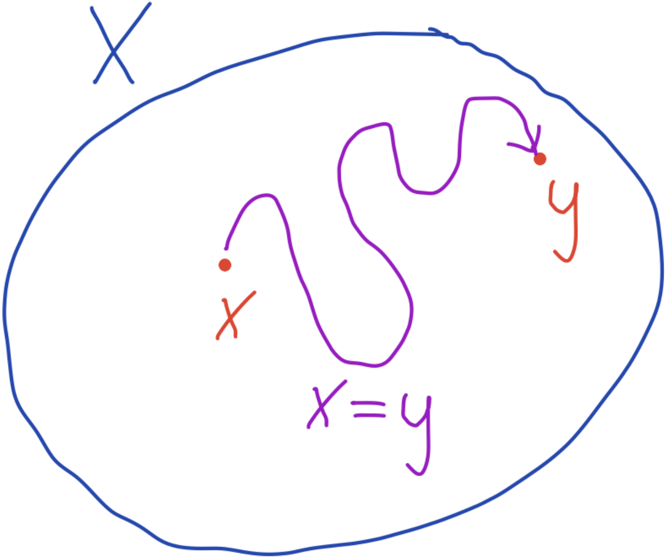

Sameness and Isomorphism in Univalent Foundations

In univalent foundations, **the aforementioned mismatch is rectified**[2]

- Groups are equal if and only if they are isomorphic
- Groups have the same properties if and only if they are isomorphic

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Sameness and Isomorphism in Univalent Foundations

In univalent foundations, **the aforementioned mismatch is rectified**[2]

- Groups are equal if and only if they are isomorphic
- Groups have the same properties if and only if they are isomorphic
- The **univalence axiom** guarantees that algebraic structures are identified up to isomorphism

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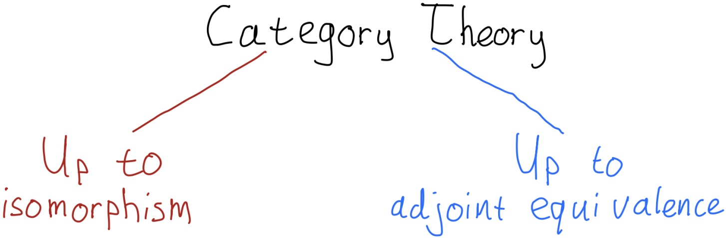
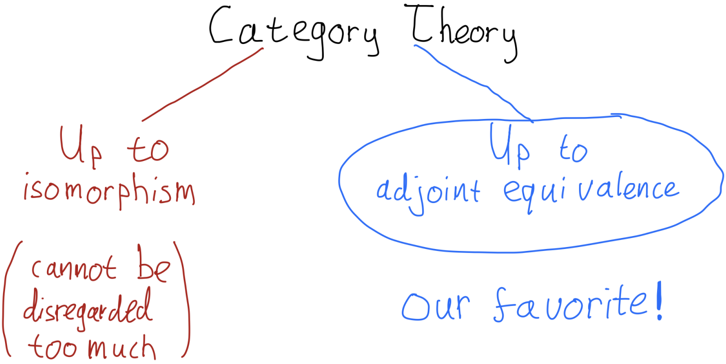
Sameness for Categorical Structures

- However, sameness is more interesting for **categorical structures**
- For instance, categories, monoidal categories, enriched categories, . . .
- Such structures can be identified up to **multiple notions of sameness**, namely **isomorphism** and **adjoint equivalence**

Two Flavors of Category Theory



Two Flavors of Category Theory



Two Kinds of Categories in Univalent Foundations

In univalent foundations, we have **two notions of categories**[3]

- **Univalent categories** who are identified up to adjoint equivalence. Their objects are identified up to isomorphism
- **Setcategories** who are identified up to isomorphism. Their objects are identified up to equality

[3] "Univalent categories and the Rezk completion" by Ahrens, Kapulkin, and Shulman

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In UF: most categories in nature are univalent
However, sometimes we meet setcategories

[3] "Univalent categories and the Rezk completion" by Ahrens, Kapulkin, and Shulman

But what about higher categories?

Higher categories come with

- more interesting notions of equivalence
- weakness versus strictness

For instance, we have the following notions of equivalence

- for **2-categories and bicategories**: isomorphism, essentially surjective & local isomorphism, biequivalence
- for **double categories**: isomorphism, vertical equivalence, gregarious equivalence

Our Philosophy

We introduce the **univalence maxim**:

"For each notion of equivalence of a given categorical structure, there exists a tailored definition whose notion of equality in univalent foundation precisely coincides with the chosen notion of equivalence"

Our Paper

Our paper: apply the univalence maxim to double categories

- In mathematical practice, double categories are viewed under multiple notions of equivalence
- Specifically, we designed notions of **univalent double category** corresponding to these notions of equivalence.
- We also **formalized** these notions using the Coq/Rocq proof assistant and the UniMath library

Our Paper

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- In mathematical practice, double categories are viewed under multiple notions of equivalence
- Specifically, we designed notions of **univalent double category** corresponding to these notions of equivalence.
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In the remainder of the talk, I will discuss

- Preliminaries: what are univalent categories?
- Preliminaries: what are double categories?

Categories in Univalent Foundations

Definition

A type X is called a **set** if for all $x, y : X$ and $p, q : x = y$, we have $p = q$.

Definition

A category is called a **setcategory** if its type of objects is a set.

Definition

A category is called **univalent** if the map from identities $x = y$ to isomorphisms $x \cong y$ is an equivalence of types.

What Are Double Categories?

A double category is given by

- objects
- horizontal morphisms
- vertical morphisms
- squares

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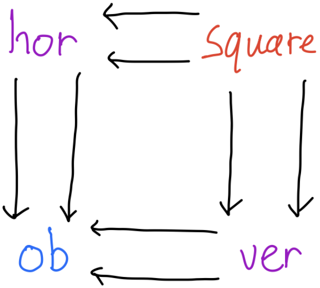
- vertical morphisms
- squares

Note:

- We have **identity and composition operations** for vertical and horizontal morphisms, and for squares
- Composition for morphisms could either be **strictly** unital and associative or **weakly**
- So: double categories come with **various notions of strictness**

What Are Double Categories?



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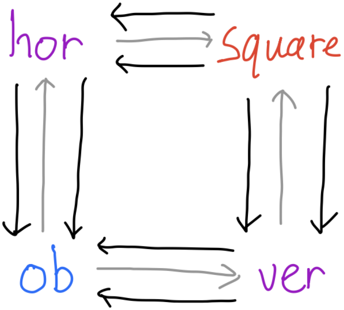

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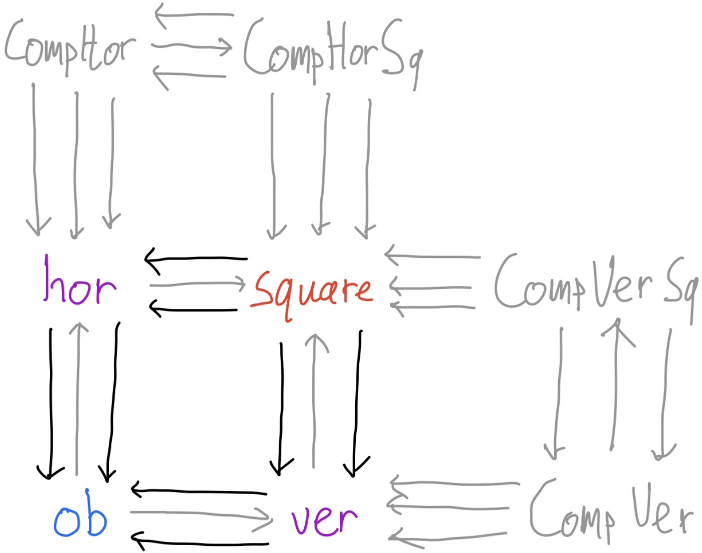


Flavors of Double Categories

There are different flavors of double categories:

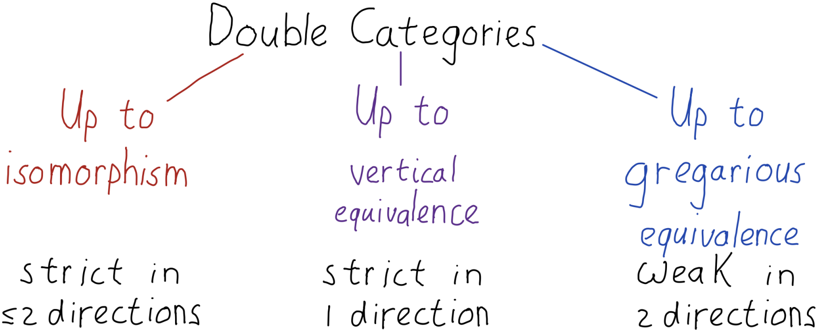
- Strict double categories: strict in both directions
- Pseudo double categories: weak in 1 direction
- Weak double categories: weak in both directions

Examples of Double Categories

	Objects	Horizontal	Vertical	Kind
Rel	sets	functions	relations	strict
Span($\mathcal{C}$)	$x, y : \mathcal{C}$	$x \rightarrow y$	spans in $\mathcal{C}$	pseudo
$\mathrm{Prof}_{\mathrm{set}}$	setcategories	functors	profunctors	pseudo
$\mathrm{Prof}_{\mathrm{univ}}$	univalent cats	functors	profunctors	weak
Sq($\mathcal{B}$)	objects in $\mathcal{B}$	1-cells	1-cells	weak

Here $\mathcal{C}$ is a category with pullbacks and $\mathcal{B}$ is a bicategory

Sameness of Double Categories



Univalent Double Categories

Definition

A pseudo double category is said to be **univalent**[4] if

- the map from identities $x = y$ of objects to vertical isomorphisms $x \cong y$ is an equivalence
- the map from identities $f = g$ of horizontal morphisms to invertible squares $f \cong g$ is an equivalence

[4] "Univalent Double Categories" by Van der Weide, Rasekh, Ahrens, North

Univalent Weak Double Categories

Definition

A weak double category is said to be **univalent** if

- the map from identities $x = y$ of objects to gregarious equivalences $x \cong y$ is an equivalence
- the map from identities $f = g$ of horizontal morphisms to invertible squares $f \cong g$ is an equivalence
- the map from identities $f = g$ of vertical morphisms to invertible squares $f \cong g$ is an equivalence

Gregarious equivalence: **symmetric** notion of equivalence for objects in a double category (involves both horizontal and vertical morphisms)

[5] "Enriched categories, internal categories and change of base" by Verity

Univalent Weak Double Categories

Definition

A weak double category is said to be **univalent** if

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- the map from identities $f = g$ of horizontal morphisms to invertible squares $f \cong g$ is an equivalence
- the map from identities $f = g$ of vertical morphisms to invertible squares $f \cong g$ is an equivalence

Gregarious equivalence: **symmetric** notion of equivalence for objects in a double category (involves both horizontal and vertical morphisms)

Unifying framework for univalent double categories: **Verity double bicategories**[5]

[5] "Enriched categories, internal categories and change of base" by Verity

Conclusion

- Higher categories come with various notions of equivalence
- **Univalence maxim**: for every flavor of equivalence of structured categories, find a suitable notion whose identity corresponds to those equivalences
- We applied this to double categories
- **Future research**: general statement for the univalence maxim using the framework by Ahrens, North, Shulman, and Tsementzis[6]

Our paper is here: `https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2025.45`

[6] "The Univalence Principle" by Ahrens, North, Shulman, Tsementzis