The Univalence Maxim and Univalent Double Categories

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Sameness for Structures

Mathematical structures often are identified up to **isomorphism**

- For instance, we identify groups up to isomorphism
- We can replace isomorphic groups by each other (as if they were equal)
- All properties that we care about in group theory, are **invariant under isomorphism**

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- All properties that **we care about in group theory**, are **invariant under isomorphism**

A Case against Set Theoretic Equality

In set theory there is a **mismatch** regarding sameness of structures

- According to the foundations: **equality**
- But in practice: **isomorphism**

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- According to the foundations: **equality**
- But in practice: **isomorphism**

Disadvantage:

- **we** need to carve out a collection of relevant properties
- and **we** need to prove that these are invariant under isomorphism

Our foundations should do that for us

Univalent Foundations

- **Univalent foundations** (UF) is a flavor of intensional Martin-Löf type theory with the **univalence axiom**[1]
- Univalence axiom: **types are identified if they are equivalent**

[1] "Homotopy type theory: Univalent foundations of mathematics"

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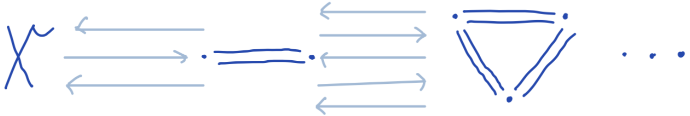
Key features of univalent foundations:

- **Identity is proof relevant**: not every proof of $p = q$ is the same! Proofs of identity carry information
- Interpretation in **homotopy theory**: types as spaces, terms as points, proofs of identity as paths

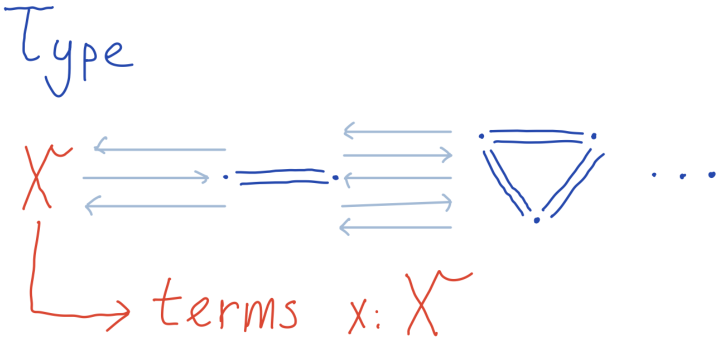
[1] "Homotopy type theory: Univalent foundations of mathematics"

Types as Spaces

Type



Types as Spaces



Types as Spaces

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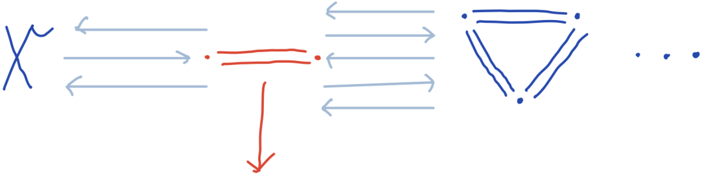
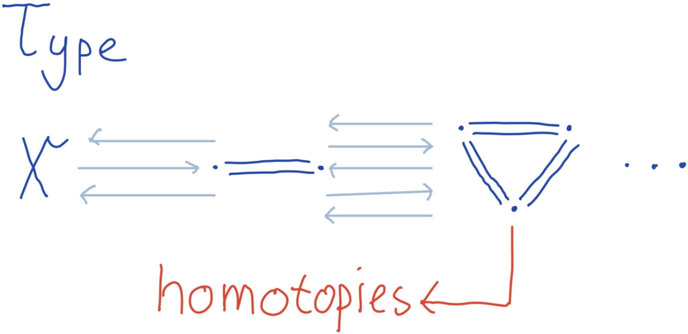


paths $p : x =_X y$

Types as Spaces



Sameness and Isomorphism in Univalent Foundations

In univalent foundations, **the aforementioned mismatch is rectified**[2]

- Groups are equal if and only if they are isomorphic
- Groups have the same properties if and only if they are isomorphic

[2] "Isomorphism is equality" by Coquand and Danielsson

Sameness and Isomorphism in Univalent Foundations

In univalent foundations, **the aforementioned mismatch is rectified**[2]

- Groups are equal if and only if they are isomorphic
- Groups have the same properties if and only if they are isomorphic
- The **univalence axiom** guarantees that algebraic structures are identified up to isomorphism

[2] "Isomorphism is equality" by Coquand and Danielsson

Sameness for Categorical Structures

- Sameness is more interesting for **categorical structures**
- For instance, categories, monoidal categories, enriched categories, ...
- Such structures can be identified up to **multiple notions of sameness**, namely **isomorphism** and **adjoint equivalence**

Two Kinds of Categories in Univalent Foundations

In univalent foundations, we have **two notions of categories**[3]

- **Univalent categories** which are identified up to adjoint equivalence. Their objects are identified up to isomorphism
- **Setcategories** which are identified up to isomorphism. Their objects are identified up to equality

[3] "Univalent categories and the Rezk completion" by Ahrens, Kapulkin, and Shulman

But what about higher categories?

Higher categories come with

- more interesting notions of equivalence
- weakness versus strictness

For instance, we have the following notions of equivalence

- for **2-categories and bicategories**: isomorphism, essentially surjective & local isomorphism, biequivalence
- for **double categories**: isomorphism, vertical equivalence, gregarious equivalence

The Univalence Maxim

We introduce the **univalence maxim**:

"For each notion of equivalence of a given categorical structure, there exists a tailored definition whose notion of equality in univalent foundation precisely coincides with the chosen notion of equivalence"

This talk

This talk: apply the univalence maxim to double categories

- In mathematical practice, double categories are viewed under multiple notions of equivalence
- Specifically, we designed notions of **univalent double category** corresponding to these notions of equivalence.
- We also **formalized** these notions using the Coq/Rocq proof assistant and the UniMath library

Categories in Univalent Foundations

Definition

A type X is called a **set** if for all $x, y : X$ and $p, q : x = y$, we have $p = q$.

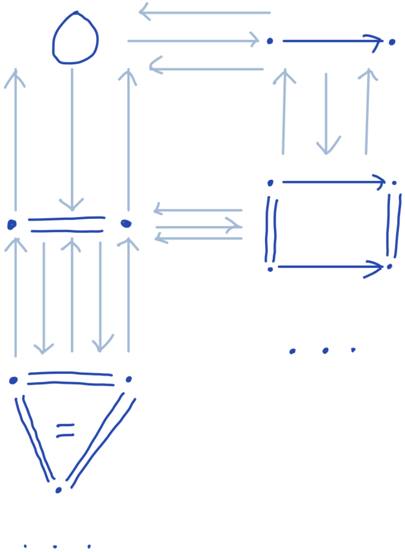
Definition

A **category** consists of

- a type O
- for each $x, y : O$ a **set** $x \to y$

with the usual identity and composition operations.

Categories in the Simplicial Sets Model



Setcategories

Definition

A category is called a **setcategory** if its type of objects is a set.

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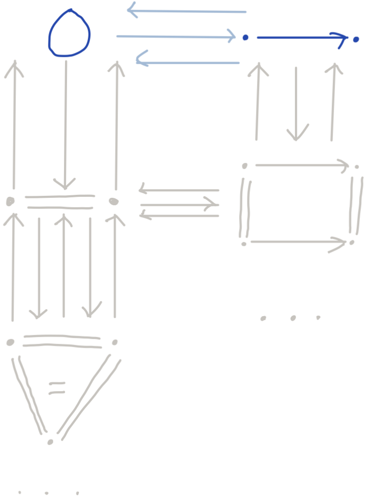
Theorem

Identity of setcategories corresponds to isomorphism.

Setcategories in the Simplicial Sets Model



Univalent Categories

Definition

A category is called **univalent** if the map from identities $x = y$ to isomorphisms $x \cong y$ is an equivalence of types.

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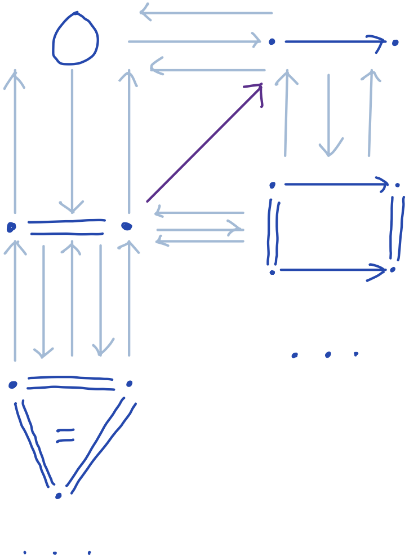
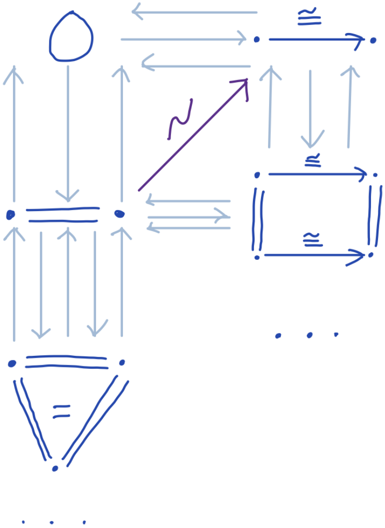
Theorem

Identity of univalent categories corresponds to adjoint equivalence.

Univalent Categories in the Simplicial Sets Model



Univalent Categories in the Simplicial Sets Model



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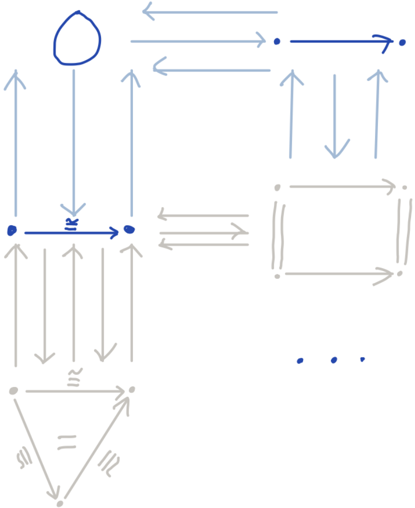


Note:

- A large part of category theory is neutral with respect to univalence versus strictness
- Specifically, one can develop it using categories

However, there are constructions where strictness versus univalence is essential

- Kleisli category
- Karoubi envelope

Bicategories

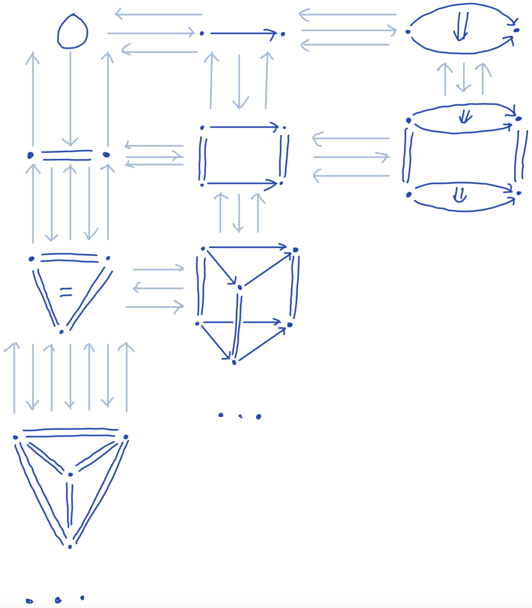
Definition

A **bicategory** consists of

- a type O of objects
- for all $x, y : O$ a type $x \to y$ of 1-cells
- for all $f, g : x \to y$ a **set** $f \Rightarrow g$ of 2-cells

We also require the usual identity and composition operations.
In addition, composition of 1-cells is weakly unital and associative (i.e., up to an invertible 2-cell).

Bicategories in the Simplicial Sets Model



Equivalence of Bicategories

Bicategories come with multiple notions of equivalence

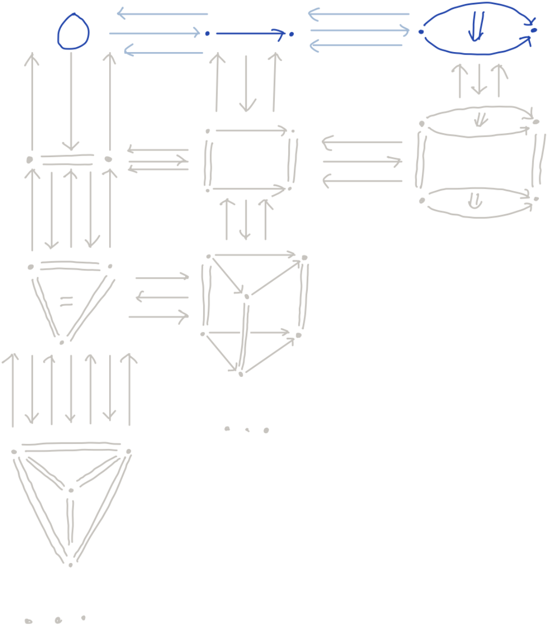
- Isomorphism
- Essentially surjective & local isomorphism
- Biequivalence

Setbicategories

Definition
A **setbicategory** is a bicategory whose types of objects and of morphisms are sets.

Theorem
Identity of setbicategories correspond to isomorphism of bicategories.

Setbicategories in the Simplicial Sets Model



Univalent 2-Categories

Definition

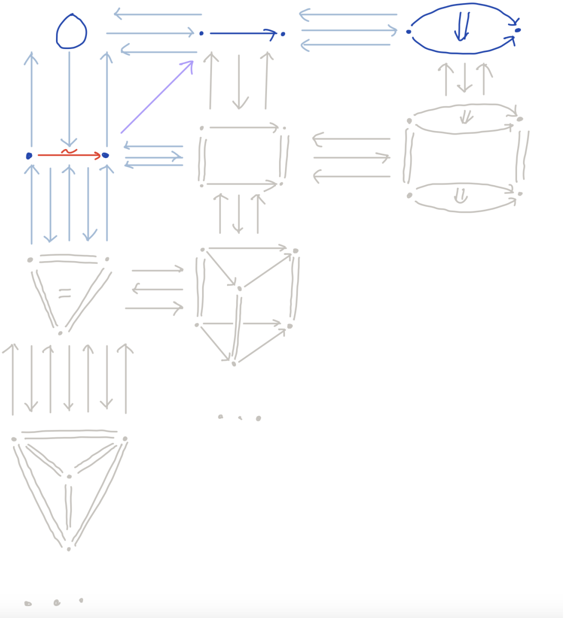
A **univalent 2-category** is a category enriched in the category of setcategories.

Note: each univalent 2-category gives rise to a bicategory

Theorem

Identity of univalent 2-categories corresponds to essentially surjective pseudofunctors that are local isomorphisms.

Univalent 2-Categories in the Simplicial Sets Model



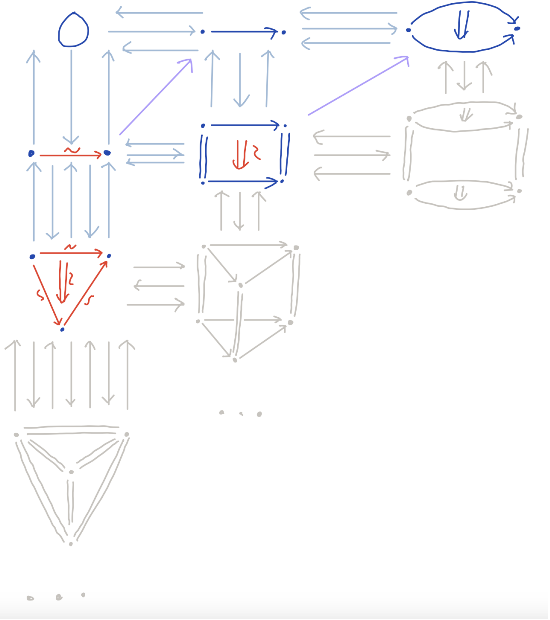
Univalent Bicategories

Definition

We say that a bicategory is

- **locally univalent** if the map from identities $f = g$ to invertible 2-cells $f \cong g$ is an equivalence of types for all $f, g : x \to y$
- **globally univalent** if the map from identities $x = y$ to adjoint equivalences $x \simeq y$ is an equivalence of types for all objects x and y
- **univalent** if it is both locally and globally univalent

Univalent Bicategories in the Simplicial Sets Model



What Are Double Categories?

A double category is given by

- objects
- horizontal morphisms
- vertical morphisms
- squares

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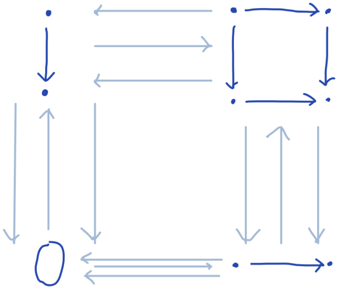
- vertical morphisms
- squares

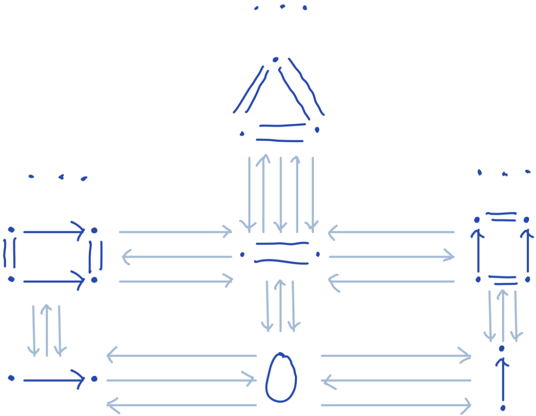
Note:

- We have **identity and composition operations** for vertical and horizontal morphisms, and for squares
- Composition for morphisms could either be **strictly** unital and associative or **weakly**
- So: double categories come with **various notions of strictness**

What Are Double Categories?



Double Categories in the Simplicial Sets Model



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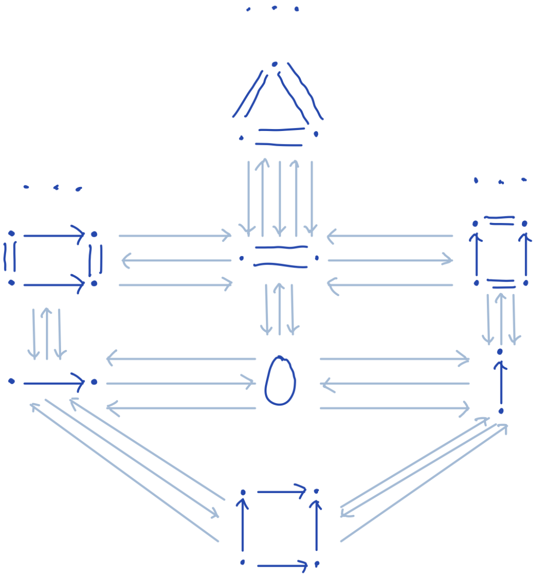

Flavors of Double Categories

There are different flavors of double categories:

- Strict double categories: strict in both directions
- Pseudo double categories: weak in 1 direction
- Weak double categories: weak in both directions

Examples of Double Categories

	Objects	Horizontal	Vertical	Kind
Rel	sets	functions	relations	strict
$\mathsf{Span}(\mathcal{C})$	$x, y : \mathcal{C}$	$x \to y$	spans in $\mathcal{C}$	pseudo
$\mathsf{Prof}_{\mathsf{set}}$	setcategories	functors	profunctors	pseudo
$\mathsf{Prof}_{\mathsf{univ}}$	univalent cats	functors	profunctors	weak
$\mathsf{Sq}(\mathcal{B})$	objects in $\mathcal{B}$	1-cells	1-cells	weak

Here $\mathcal{C}$ is a category with pullbacks and $\mathcal{B}$ is a bicategory

Sameness of Double Categories

Double categories come with multiple notions of equivalence

- Isomorphism
- Horizontal equivalence
- Gregarious equivalence

Strict Double Categories

Definition

A **strict double setcategory** is a double category such that

- the types of objects, of horizontal morphisms, and of vertical morphisms are sets
- composition is strictly unital and associative in both directions

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Strict double categories are identified up to isomorphism

Univalent Double Categories

Definition

A **univalent pseudo double category** is a double category such that[4] if

- vertical composition is strictly unital and associative
- horizontal composition is weakly unital and associative
- the map from identities $x = y$ of objects to vertical isomorphisms $x \cong y$ is an equivalence
- the map from identities $f = g$ of horizontal morphisms to invertible squares $f \cong g$ is an equivalence

[4] "Univalent Double Categories" by Van der Weide, Rasekh, Ahrens, North

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Univalent pseudo double categories are identified up to vertical equivalence

[4] "Univalent Double Categories" by Van der Weide, Rasekh, Ahrens, North

Gregarious Equivalences

Gregarious equivalence give us a symmteric notion of equivalence between objects.

Definition

A **gregarious equivalence** between objects x and y in a double category consist of

- a horizontal adjoint equivalence h from x to y
- a vertical adjoint equivalence v from x to y

such that h and v form a companion pair.

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such that h and v form a companion pair.

Definition

A **gregarious equivalence of double categories** is a functor of double categories that is surjective up to gregarious equivalence, full on horizontal and vertical 1-cells up to globular invertible square, fully faithful

Univalent Weak Double Categories

Definition

A **univalent weak double category** is double category such that

- vertical and horizontal composition are weakly unital and associative
- the map from identities $x = y$ of objects to gregarious equivalences $x \cong y$ is an equivalence
- the map from identities $f = g$ of horizontal morphisms to invertible squares $f \cong g$ is an equivalence
- the map from identities $f = g$ of vertical morphisms to invertible squares $f \cong g$ is an equivalence

Univalent Weak Double Categories

Definition

A **univalent weak double category** is double category such that

- vertical and horizontal composition are weakly unital and associative
- the map from identities $x = y$ of objects to gregarious equivalences $x \simeq y$ is an equivalence
- the map from identities $f = g$ of horizontal morphisms to invertible squares $f \cong g$ is an equivalence
- the map from identities $f = g$ of vertical morphisms to invertible squares $f \cong g$ is an equivalence

Univalent weak double categories are identified up to gregarious equivalence

Introduction

Univalence Maxim for Categories

Univalence Maxim for Bicategories

Univalence Maxim for Double Categories

Conclusion

Summary

- Higher categories come with various notions of equivalence
- **Univalence maxim**: for every flavor of equivalence of structured categories, find a suitable notion whose identity corresponds to those equivalences
- We applied this to categories, bicategories, double categories
- We defined suitable notions of univalence and univalence principles

A General Perspective on the Univalence Maxim?

Ahrens, North, Shulman, and Tsementzis[5] gave a general framework for univalence principles.
In their framework, one describes structures as follows

$$
\begin{tikzcd}
T \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & I \arrow[d] & E \arrow[dl, shift left] \arrow[dl, shift right] \\
& A \arrow[d, shift left] \arrow[d, shift right] & \\
& O &
\end{tikzcd}
$$

Examples: categories, bicategories, Verity double bicategories
Future work: general statement of the univalent maxim in this framework

[5] "The Univalence Principle" by Ahrens, North, Shulman, Tsementzis

Companion Pairs

A horizontal morphism $h : x \to y$ and a vertical morphism $v : x \to y$ form a **companion pair** if there are squares

$$
\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\eta"] & x \arrow[d, "v"] \\
x \arrow[r, "h"'] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\varepsilon"] & x \arrow[d, "v"] \\
x \arrow[r, "h"'] & y
\end{tikzcd}
$$

such that the following squares are identities

$$
\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "\eta\Downarrow"] & x \arrow[d, "v"] \\
x \arrow[r, "h"] \arrow[d, "v"'] \arrow[dr, phantom, "\varepsilon\Downarrow"] & y \arrow[d] \\
y \arrow[r] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\eta"] & x \arrow[r, "h"] \arrow[d, "v"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & y \arrow[d] \\
x \arrow[r, "h"'] & y \arrow[r] & y
\end{tikzcd}
$$