Impredicative Encodings of Inductive and Coinductive Types

Steven Bronsveld, Herman Geuvers, **Niels van der Weide**

Impredicative Encodings

- **Impredicative encodings** allow us to reduce inductive types to elementary type formers: $\prod$, $\rightarrow$
- This is how one would implement them in Rocq in the past
- Only suitable in impredicative settings

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Impredicativity: we have an impredicative universe $\mathcal{U}$ closed under $=$ and $\sum$ and the following rule

$$\frac{\Gamma \vdash A\ \mathrm{Type} \qquad \Gamma, x : A \vdash B\ x : \mathcal{U}}{\Gamma \vdash \prod(x : A), B\ x : \mathcal{U}}$$

Impredicative Encoding of Lists

Let E be a type. Define $\mathsf{List}^* : \mathcal{U}$ as follows.

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$$\mathsf{cons}^*\ e\ l = \lambda(X : \mathcal{U})(n : X)(c : E \to X \to X), c\ e\ l$$

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$$\mathsf{rec}_{\mathsf{List}^*} : \prod (X : \mathcal{U}), X \to (E \to X \to X) \to \mathsf{List}^* \to X$$

$$\mathsf{rec}_{\mathsf{List}^*}\ X\ n\ c = \lambda (l : \mathsf{List}^*), l\ X\ n\ c$$

But.....

- What do we want of inductive types? **Induction principles!**
- For List*, we can prove the **recursion principle** with the expected β-rules
- However, **induction is not derivable**[1]

List* is not an initial algebra, uniqueness does not hold in general.

[1] Geuvers, "Induction is not derivable in second order dependent type theory"

Fixing Impredicative Encodings

Awodey, Frey, and Speight: don't worry, we can fix this [2]

- Intuition: the type List^* has "too many inhabitants"
- Define a predicate $\mathsf{Lim}_{\mathsf{List}}$ on List^* (next slide)
- Define List to be $\sum(l : \mathsf{List}^*), \mathsf{Lim}_{\mathsf{List}}\ l$

[2] Awodey, Frey, Speight, "Impredicative encodings of (higher) inductive types"

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- Define a predicate $\mathsf{Lim}_{\mathsf{List}}$ on List^* (next slide)
- Define List to be $\sum(l : \mathsf{List}^*), \mathsf{Lim}_{\mathsf{List}}\ l$
- One can prove that List is an initial algebra
- Initial algebra semantics: List satisfies induction

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Fixing Impredicative Encodings

To define $\mathrm{Lim}_{\mathrm{List}}$:
Suppose we have a commuting square.

$$\begin{tikzcd} 1 + E \times X \arrow[r, "\mathrm{id} \times f"] \arrow[d, "{[n_X, c_X]}"'] & 1 + E \times Y \arrow[d, "{[n_Y, c_Y]}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

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X \arrow[rr, "f"] & & Y \\
& \mathsf{List}^* \arrow[ul, "\mathsf{rec}_{\mathsf{List}^*}\ X\ n_X\ c_X"] \arrow[ur, "\mathsf{rec}_{\mathsf{List}^*}\ Y\ n_Y\ c_Y"'] &
\end{tikzcd}$$

Then the bottom triangle must commute.

Fixing Impredicative Encodings

We say that $l : \mathsf{List}^*$ satisfies $\mathsf{Lim}_{\mathsf{List}}$ if for all

- $X : \mathcal{U}$ together with $n_X : X$, $c_X : E \to X \to X$
- $Y : \mathcal{U}$ together with $n_Y : Y$, $c_Y : E \to Y \to Y$

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- $f : X \to Y$
- $p_n : f\ n_X = n_Y$
- $p_c : \prod(e : E)(x : X), f\ (c_X\ e\ X) = c_Y\ e\ (f\ x)$

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we have

$$f\ (\mathsf{rec}_{\mathsf{List}^*}\ X\ n_X\ c_X\ l) = \mathsf{rec}_{\mathsf{List}^*}\ Y\ n_Y\ c_Y\ l$$

Other Encodings

Awodey, Frey, and Speight considered

- sum types
- algebras for a functor on sets (i.e., types for which there's at most one proof that $x = y$)
- natural numbers
- the circle

They worked in a setting without uniqueness of identity proofs

[3] Echeveste, "Alternative impredicative encodings of inductive types"

[4] `https://homotopytypetheory.org/2018/11/26/impredicative-encodings-part-3/`

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Note: one can get rid of the truncation assumption[3] [4]

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This work: coinductive types

We look at the dualization

- define coinductive types using impredicative encodings
- prove suitable coinduction principles, i.e., bisimulation corresponds to equality

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- define coinductive types using impredicative encodings
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This talk:

- we demonstrate the method for streams
- we discuss how to extend it to M-types

Main Idea

Recall:

$$\mathsf{List}^* = \prod(X : \mathcal{U}), X \to (E \to X \to X) \to X$$

$$\mathsf{List} = \sum(l : \mathsf{List}^*), \mathsf{Lim}_{\mathsf{List}}\ l$$

To dualize this construction:

- To dualize $\prod$, we use **existential types**
- To dualize the subtype: we use **quotient types**

Main Idea

Recall:

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- To dualize $\prod$, we use **existential types**
- To dualize the subtype: we use **quotient types**

We define existential types and quotient types via impredicative encodings.

Existential Types

Let $P : \mathcal{U} \to \mathcal{U}$ be a type family. Then we have

- $\exists (X : \mathcal{U}), P\, X : \mathcal{U}$
- pack : $\prod (X : \mathcal{U}), P\, X \to \exists (X : \mathcal{U}), P\, X$

Existential Types

Let $P : \mathcal{U} \to \mathcal{U}$ be a type family. Then we have

- $\exists (X : \mathcal{U}), P\,X : \mathcal{U}$
- $\mathsf{pack} : \prod (X : \mathcal{U}), P\,X \to \exists (X : \mathcal{U}), P\,X$

together with a recursion principle:

$$\begin{aligned}\mathsf{rec}_{\exists} : \prod (Y : \mathcal{U}),\\ &(\prod (Z : \mathcal{U}), P\,Z \to Y)\\ &\to (\exists (X : \mathcal{U}), P\,X)\\ &\to Y\end{aligned}$$

satisfying the expected β- and η-rules.

Existential Types

Impredicative encoding: we define $\exists^*(X : \mathcal{U}), P\,X$ to be

$$\prod(Y : \mathcal{U}), (\prod(Z : \mathcal{U}), (P\,Z \to Y) \to Y) \to Y$$

We define $\mathsf{Lim}_\exists$ similarly to $\mathsf{Lim}_{\mathsf{List}}$ and

$$\exists(X : \mathcal{U}), P\,X = \sum(x : \exists^*(X : \mathcal{U}), P\,X), \mathsf{Lim}_\exists\,x$$

Encoding Streams

Let E be a type. We define Stream^* as follows[5].

$$\text{Stream}^* = \exists (X : \mathcal{U}), X \times (X \to E) \times (X \to X)$$

This allows us to define:

- $\text{hd}^* : \text{Stream}^* \to E$
- $\text{tl}^* : \text{Stream}^* \to \text{Stream}^*$
- $\text{corec} : \prod (X : \mathcal{U}), (X \to E) \to (X \to X) \to X \to \text{Stream}^*$

[5] Geuvers. "The Church-Scott representation of inductive and coinductive data"

Encoding Streams: Tails

Let's see how to define $\mathsf{tl}^* : \mathsf{Stream}^* \to \mathsf{Stream}^*$.

$$\mathsf{tl}^*\ s = ?$$

where $? : \mathsf{Stream}^*$

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Encoding Streams: Tails

Let's see how to define $\mathrm{tl}^* : \mathrm{Stream}^* \to \mathrm{Stream}^*$.

$$\mathrm{tl}^*\, s = \mathrm{rec}_{\exists}\, \mathrm{Stream}^*\, (\lambda Z\, z\, h\, t, ?)\, s$$

where $? : \mathrm{Stream}^*$
Here:

- $Z : \mathcal{U}$
- $z : Z$
- $h : Z \to E$
- $t : Z \to Z$

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$$\mathsf{tl}^*\, s = \mathsf{rec}_\exists\, \mathsf{Stream}^*\, (\lambda Z\, z\, h\, t, \mathsf{pack}\, Z\, ?)\, s$$

where $? : Z \times (Z \to E) \times (Z \to Z)$
Here:

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Fixing the Impredicative Encoding for Streams

- Just like for lists, we cannot prove a suitable coinduction principle for Stream^*.
- Fix for lists: take a subtype
- Fix for streams: take a **quotient**

Quotient Types

Using impredicative encodings, we construct quotient types
Let $X : \mathcal{U}$ and let $R : X \to X \to \mathcal{U}$ be a relation. Then we have

- a type $X/R : \mathcal{U}$
- a function $\mathsf{cls} : X \to X/R$

For all $Y : \mathcal{U}$ and $f : X \to Y$ that respects R, there is a unique $\mathsf{rec}_{\mathsf{Q}}\ Y\ f$ making the following diagram commute

$$\begin{tikzcd} X \arrow[r, "\mathsf{cls}"] \arrow[d, "f"'] & X/R \arrow[dl, dashed, "\mathsf{rec}_{\mathsf{Q}}\ Y\ f"] \\ Y & \end{tikzcd}$$

Quotient Types

The starting point is the following type:

$$X/^*R = \prod(Z : \mathcal{U})(f : X \to Z), \mathsf{resp}\ f\ R \to Z$$

Here $\mathsf{resp}\ f\ R$ says that f respects R.

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We define $\mathsf{Lim}_{\mathsf{Q}}$ similarly to $\mathsf{Lim}_{\mathsf{List}}$ and

$$X/R = \sum(x : X/^*R), \mathsf{Lim}_{\mathsf{Q}}\ x$$

Recall: Fixing Impredicative Encodings for Lists

To define $\mathsf{Lim}_{\mathsf{List}}$:
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Then the bottom triangle must commute.

Recall: Fixing Impredicative Encodings for Streams

Suppose we have a commuting square.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "{[h_X, t_X]}"'] & Y \arrow[d, "{[h_Y, t_Y]}"] \\
E \times X \arrow[r, "\mathrm{id} \times f"] & E \times Y
\end{tikzcd}$$

Fixing Impredicative Encodings for Streams

Suppose we have a commuting square.

$$
\begin{tikzcd}
 & \mathrm{Stream}^* & \\
X \arrow[ur, "\mathrm{corec}\ X\ h_X\ t_X"] \arrow[rr, "f"] \arrow[d, "{[h_X,t_X]}"'] & & Y \arrow[ul, "\mathrm{corec}\ Y\ h_Y\ t_Y"'] \arrow[d, "{[h_Y,t_Y]}"] \\
E \times X \arrow[rr, "\mathrm{id}\times f"] & & E \times Y
\end{tikzcd}
$$

Then the upper triangle must commute

Fixing Impredicative Encodings for Streams

Given $\sigma, \tau : \mathsf{Stream}^*$, we say $\sigma \equiv \tau$ if

$$\begin{aligned} \exists &(X : \mathcal{U})(h_X : X \to E)(t_X : X \to X) \\ &(Y : \mathcal{U})(h_Y : Y \to E)(t_Y : Y \to Y) \end{aligned}$$

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& (x : X), \\
& \sigma = \mathrm{corec}\, X\, h_X\, t_X\, x \\
& \wedge \tau = \mathrm{corec}\, Y\, h_Y\, t_Y\, (f\, x)
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Define Stream = Stream$^*/\equiv$.

Extending to M-types

- Recall: M-types are final coalgebras for polynomial functors"

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- Recall: M-types are final coalgebras for polynomial functors"

$$P(X) = \sum(a : A), B\ a \to X$$

- For M-types, we can take the same steps
- We first $M^*(A, B)$ using existential types
- Then we define a relation $\equiv$ on $M^*(A, B)$
- We define $M(A, B)$ as the quotient $M^*(A, B)/\equiv$

The Impredicative Encoding for M-types

Let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$. Define

$$M^*(A, B) = \exists (X : \mathcal{U}), X \times \left(X \to \sum (a : A), B\ a \to X\right)$$

Introduction

Streams via Imprecative Encodings

Extension to M-Types

Conclusion

Conclusion

Key points:

- We can use impredicative encodings to define inductive and coinductive types
- For inductive types: use a subtype (Awodey, Frey, Speight)
- Dual for coinductive types: use existential and quotient types
- This method works for M-types