The Interval Domain in Homotopy Type Theory*

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Abstract. Even though the real numbers are the cornerstone of many fields in mathematics, it is challenging to formalize them in a constructive setting, and in particular, homotopy type theory. Several approaches have been established to define the real numbers, and the most prominent of them are based on Dedekind cuts and on Cauchy sequences. In this paper, we study a different approach towards defining the real numbers. Our approach is based on domain theory, and in particular, the interval domain, and we build forth on recent work on domain theory in univalent foundations. All the results in this paper have been formalized in Coq as part of the UniMath library.

1 Introduction

The real numbers are one of the basic objects in mathematics, with a wide variety of applications ranging from geometry to probablistic programming. Classically, there are many established approaches for defining the reals. However, formalizing real numbers in a constructive setting remains a challenging task.

There are two causes for this difficulty. Firstly, there are many different representations of real numbers, such as Dedekind cuts and Cauchy sequences (see the work of Geuvers et al. [23] for discussion of some representations). These two representations are equivalent to each other in a classical setting, and then it does not matter which one is used. However, this equivalence depends on the axiom of countable choice, meaning that these two representations are not equivalent in a constructive setting.

Depending on the precise foundations, Dedekind cuts and Cauchy sequences have their own drawbacks. Since Dedekind cuts are defined using power sets, they raise the universe level in a predicative setting. If one assumes impredicativity (e.g., propositional resizing), then one can avoid going to a larger universe. For Cauchy reals, on the other hand, one needs to use either quotients or setoids to guarantee that the equality relation is correct. As a consequence, one either has

* This paper is dedicated to our supervisor and teacher, Herman Geuvers, on the occasion of his birthday. Herman, you teachings and research in domain theory, type theory, and mechanized proofs have greatly inspired us, and we hope that you enjoy this little expedition in the world of constructive real numbers, combining all the topics above.

to assume the axiom of countable choice to tame the quotients, or one has to manually deal with setoids. Note that if one assumes quotient inductive-inductive types, then one can avoid both of these [25].

Another challenge comes from defining the division operation on the real numbers. Since we cannot divide by zero, division is a partial operation: its input consists of a numerator, a denominator, and a proof that the denominator is distinct from 0. To define this operation constructively, the proof that the denominator is non-zero should be positive, because this allows us to extract a suitable approximation. For this reason, constructive versions of the notion of fields make use of *apartness relations*.

In this paper we study a constructive formalization of the (Dedekind) real numbers based on the ideas from domain theory. This approach, inspired by recent work of De Jong [17,18], allows us to formulate real numbers without setoids, quotients, or higher inductive-inductive types, and gives us an easier treatment of apartness, by appealing to generic domain theoretic principles. For example, the fact that arithmetic operations are strongly extensional (i.e. they reflect apartness) follow from general facts of domain theory. The main result is a construction of an ordered field of real numbers in univalent foundations. While this theorem is not new, the method that we used to prove it, is new.

More specifically, we construct a domain $\mathbf{I}\mathbb{R}$ of *interval reals* (sometimes referred to as "partial real numbers"), and define real numbers to be the maximal elements of this domain. Classically, the domain $\mathbf{I}\mathbb{R}$ contains intervals $[x, y] \subseteq \mathbb{R}$ of real numbers ordered by reverse inclusion. From the computational point of view, an interval $[x, y]$ represents a computation of a real number that only results in an approximate information, that the real number lies in the interval $[x, y]$. The higher we go in the domain, the more information we get, and the smaller the interval becomes. The maximal elements of this domain are then the singleton intervals $[x, x]$ representing exact real numbers.

Since our goal is to construct the real numbers from the interval domain, there are slight differences in our approach. If we would use the definition of $\mathbf{I}\mathbb{R}$ as above, then we would get a circular definition. Instead, we construct the domain $\mathbf{I}\mathbb{R}$ out of rational numbers using general techniques from domain theory. More specifically, we define $\mathbf{I}\mathbb{R}$ as the rounded ideal completion of rational intervals ordered by reverse strict inclusion. Secondly, the notion of maximality is not suitable in a constructive setting, and instead, the correct notion of maximality in a constructive setting is *strong maximality* [16,17]. For instance, if we stick with regular maximality, then proving that the maximal elements of $\mathbf{I}\mathbb{R}$ are the Dedekind reals requires weak excluded middle.

Just constructing the type of real numbers is, of course, not enough. We show that the real numbers form a constructive ordered Archimedean field. To do so, we define a number of arithmetical operations on the real numbers. Domain theory helps us with that: we can extend operations on rational intervals to the real numbers. The main challenge then lies in proving that these operations preserve strongly maximal elements (i.e. they give rise to operations on the real numbers). To help us with that task, we follow the approach of Bauer and Taylor

[5], and we identify a number of *locatedness* properties for interval reals. In addition, a constructive field comes with an apartness relation that should satisfy several properties. Here another application of domain theory arises, because every DCPO comes with an intrinsic apartness relation [17], which is well-behaved on strongly maximal elements.

Foundations. The results in this paper are formalized in Coq using the UniMath library [36]. Note that even though UniMath is based on univalent foundations, this paper is written in the language of set theory, for ease of understanding. In this paper, we also assume propositional resizing, which says that every proposition in universe level is equivalent to one in the lowest universe.

The formalization can be found online in a "frozen" state at `https://zenodo.org/doi/10.5281/zenodo.10664690`. The entry point to our results is the module `UniMath.OrderTheory.DCPOs.Examples.Reals`. Our formalization is mostly complete, with several admitted results about interval arithmetic, and we are in the proceess of merging the formalization into the upstream UniMath repository.

Synopsis. The rest of the paper is organized as follows. In Section 2 we recall the preliminaries on domain theory. After that, we construct the DCPO of interval reals in Section 3, and we define the real numbers to be the strongly maximal interval reals. In Section 4, we characterize strong maximality of interval reals via a notion of *locatedness*. In Section 5 we define arithmetic operations on $\mathbf{I}\mathbb{R}$, and we show that they restrict to operations on $\mathbb{R}$. We show that the real numbers form a constructive ordered field in Section 6. Finally, in Section 7 we conclude and discuss related work.

2 Preliminaries on Domain Theory

In this section we briefly recall the notions from domain theory needed in the remainder of the paper. Most the material here is standard (we refer an interested reader to a classical text on domain theory [1]), or, when mentioned, to the recent work of De Jong on constructive domain theory [17].

2.1 (Continuous) DCPOs

A *directed-complete partial order* (DCPO) is a set D together with a partial order $\sqsubseteq$ such that every directed subset X of D (i.e. a subset that contains upper bounds for each pair of elements) has the least upper bound (lub) $\bigsqcup X$. Morphisms between DCPOs are *Scott-continuous* functions, i.e. monotone functions that preserve the least upper bounds.

Remark 1. Type theoretically, a directed set in a partial order $(D, \sqsubseteq)$ is given by a type I and a function $f : I \to D$ such that I is inhabited (written $\|I\|$), and that $\{f(x) \mid x \in I\}$ is directed in the usual sense. Notice that here we follow De Jong and Escardó [27] and take directed sets to be inhabited. In particular, this means that DCPOs do not have to be pointed.

Given two elements $x, y \in D$ of a DCPO, we say that x is *way below* y, denoted as $x \ll y$, if for any directed set $X \subseteq D$, if $y \sqsubseteq \bigsqcup X$, then there exists some element $b \in X$ for which $x \sqsubseteq b$.

We are particularly interested in DCPOs that are "generated" by the way below relation. We say that a DCPO D is *continuous* if for every $x \in D$ the set $\{y \in D \mid y \ll x\}$ is directed and its supremum is x itself. There is a systematic way of constructing continuous DCPO out of what is called an *abstract basis*.

Definition 2. *An* ***abstract basis*** *is a set B together with a transitive relation $\prec$ that satisfies the following* interpolation property*:*

- *for each $a \in B$ there is a element $b \in B$ such that $b \prec a$;*
- *for each $a_1, a_2, b \in B$ such that $a_1, a_2 \prec b$, there is an interpolant $a \in B$ such that $a_1, a_2 \prec a \prec b$.*

Given an abstract basis $(B, \prec)$, we construct a DCPO $\mathsf{RIdl}(B, \prec)$ in which the way below relation is induced by the $\prec$ relation of the basis.

Definition 3. *A* ***rounded ideal*** *is a set $X \subseteq B$ of basis elements that is inhabited, downwards closed, and contains upper bounds: if $a_1, a_2 \in X$ then there exists some $b \in X$ such that $a_1, a_2 \prec b$.*

The ***rounded ideal completion*** *of B is defined to be the DCPO whose underlying set consists of all rounded ideals over the basis $(B, \prec)$, and whose order is given by the subset relation.*

There is a monotone map from B to $\mathsf{RIdl}(B, \prec)$ sending a basis element b to the *principal rounded ideal* $\downarrow(b) = \{a \in B \mid a \prec b\}$. There are some important facts about rounded ideal completions that we use.

Lemma 4. *Every element $X \in \mathsf{RIdl}(B, \prec)$ is the least upper bound of the basis elements included in it: $X = \bigcup\{\downarrow b \mid b \in X\}$.*

Lemma 5. *The way below relation on $\mathsf{RIdl}(B, \prec)$ is "induced" by $\prec$. In the sense that for any elements a, b of the basis we have:*

- *If $a \prec b$ then $\downarrow a \ll \downarrow b$;*
- *If $a \in X$ then $\downarrow a \ll X$;*
- *If $X \ll Y$ then there exists $b \in Y$ such that $X \subseteq \downarrow b$.*

The rounded ideal completion satisfies the universal property which allows us to lift monotone functions from the basis to the ideal completion.

Definition 6. *Let $f : B \to Y$ be a function from an abstract basis B to the DCPO Y such that for all $a, b \in B$ with $\downarrow a \subseteq \downarrow b$, we have $f(a) \sqsubseteq f(b)$. Then we extend f to a Scott-continuous function $f^* : \mathsf{RIdl}(B, \prec) \to Y$ defined as*

$$f^*(X) = \bigsqcup\{f(b) \mid b \in X\}.$$

The extension commutes with the principal ideals up to inequality, which means that $f^*(\downarrow b) \sqsubseteq f(b)$. It is the greatest function with such property. More specifically, for every Scott-continuous function g that satisfies $g(\downarrow b) \sqsubseteq f(b)$ we have $g(X) \sqsubseteq f^*(X)$ for any ideal X.

2.2 Scott Topology and Apartness

Every DCPO comes with an intrinsic notion of topology, in which the open sets are *Scott-open.* A set $X \subseteq D$ is Scott-open if it is upwards closed and whenever $\bigsqcup Y \in X$ then there is already some $y \in Y$ for which $y \in X$. This forms a topology in the usual sense, and if the DCPO D is a rounded ideal completion, then the topology is generated by the principal open sets over the principal ideals. Using the Scott topology, we define the *intrinsic apartness* relation. We say that x is apart from y, written $x \# y$, if there exists a Scott open set containing x and not containing y, or the other way around.

Intrinsic apartness is irreflexive and symmetric, but in general it is not tight or cotransitive (see [17, Theorem 58]). A stronger notion, also stemming from topology is *Hausdorff-separatedness.* The elements x and y are Hausdorff-separated if there are disjoint Scott-open sets S_1, S_2 such that $x \in S_1$ and $y \in S_2$. We have the following characterization of Hausdorff-separatedness in continuous DCPOs.

Proposition 7 ([18, Lemma 76]). *Two elements $x, y \in D$ of a continuous DCPO with a basis B are Hausdorff-separated iff there are elements $a, b \in B$ such that (i) $a \ll x$, (ii) $b \ll y$, (iii) there is no $z \in D$ such that $a \ll z$ and $b \ll z$.*

Proposition 8. *Scott continuous maps reflect apartness. More specifically, given a Scott continuous map $f : D_1 \to D_2$ and elements $x, y : D_1$ such that $f(x) \# f(y)$, we have $x \# y$.*

2.3 Strongly maximal elements.

The right notion of maximality in a constructive setting is *strong maximality.* In the context of impredicative set theory it was studied by De Jong [17], based on a notion of constructively maximal elements, studied in classical meta-theory by Smyth [34]. The work by De Jong carries over to homotopy type theory with resizing without any essential modification.

Definition 9 ([17, Definition 70]). *An element $x \in D$ of a DCPO is* ***strongly maximal*** *if for any $u, v \in D$ with $u \ll v$ either $u \ll x$ or v and x are Hausdorff-separated. If D is a continuous DCPO, then it suffices to consider the elements u, v from a basis of D.*

Proposition 10 ([17, Proposition 80]). *Strongly maximal elements in a continuous DCPO are maximal. If $x \in D$ is strongly maximal and $x \sqsubseteq y$, then $x = y$.*

Proposition 11 ([17, Proposition 85]). *The relative Scott topology on the strongly maximal elements in a continuous DCPO is Hausdorff, and the intrinsic apartness on strongly maximal elements coincides with Hausdorff-separatedness.*

Proposition 12. *The intrinsic apartness relation is tight and cotransitive on strongly maximal elements. That is to say, $\neg(x \# y)$ implies $x = y$, and $x \# y$ implies $x \# z \vee z \# y$.*

In [17], Proposition 12 is proven by showing that every strongly maximal element is sharp (see [17, Definition 59] for the precise definition), and that the corresponding statement holds for sharp elements.

3 The Interval Domain

In this section we construct the set $\mathbb{R}$ of real numbers, using the domain-theoretic techniques described in Section 2. To do so, first define the set of *open rational intervals*, and we show that it gives rise to an abstract basis in Proposition 13. We also define the arithmetical operations on rational intervals following [31], which we use in Section 5 to define arithmetic operations on real numbers. The set of *interval reals* is defined to be the rounded ideal completion of the rational intervals (Definition 14), and the *real numbers* are defined to be the strongly maximal interval reals (Definition 15).

An (open, non-empty) rational interval is a pair $I = (d, u) \in \mathbb{Q} \times \mathbb{Q}$ such that $d < u$. We write $\underline{I}$ and $\overline{I}$ for left and right endpoints of $I = (d, u)$, i.e. d and u respectively. By $\mathbf{I}\mathbb{Q}$ we denote the set of rational intervals. We write $|I|$ for the size $\overline{I} - \underline{I}$ of the interval I.

Proposition 13. *We have an abstract basis whose underlying set consists of rational intervals. The order relation of this abstract basis is given by reverse strict inclusion* $\supsetneq$*, which is defined by*

$$I \supsetneq J \iff \underline{I} < \underline{J} \wedge \overline{J} < \overline{J}.$$

To prove Proposition 13, we need to show that $(\mathbf{I}\mathbb{Q}, \supsetneq)$ satisfies the two interpolation properties in Definition 2. If we have an interval I, we must find a J such that $J \supsetneq I$. We can take J to be $(\underline{I} - 1, \overline{I} + 1)$. Furthermore, if we have intervals I_1, I_2, K such that $I_1 \supsetneq K$ and $I_2 \supsetneq K$, we need to find an interval J such that $I_1 \supsetneq J$, $I_2 \supsetneq J$, and $K \supsetneq J$. We define J as follows:

$$J = (\frac{\max\{\underline{I_1}, \underline{I_2}\} + \underline{K}}{2}, \frac{\min\{\overline{I_1}, \overline{I_2}\} + \overline{K}}{2}).$$

In Section 5, we define operations on real numbers, and for that, we need several arithmetical operations on real intervals [31]. Chiefly among them are various algebraic operations that are lifted from $\mathbb{Q}$ to $\mathbf{I}\mathbb{Q}$. Some of the operations are given in Figure 1, and the full implementations are found in the Coq source code. Not all arithmetical laws hold for the operations in Figure 1. For example, addition and multiplication on intervals is associative and commutative. However, distributivity and neutrality for addition and multiplication do not hold.

Next we define the *interval domain*. Usually, one first defines the real numbers, and then the interval domain is defined to be the collection of real intervals. One can then show that this is a continuous DCPO, and that the set of rational intervals forms a basis for this DCPO. Our definition is reversed compared to this approach: using the fact that the rational intervals form an abstract basis, we can take the rounded ideal completion to acquire a continuous DCPO with that basis.

$$I + J = (\underline{I} + \underline{J}, \overline{I} + \overline{J}) \qquad -I = (-\overline{I}, -\underline{I})$$
$$I * J = (\min P, \max P) \qquad \text{where } P = \{\underline{I}\cdot\underline{J}, \underline{I}\cdot\overline{J}, \overline{I}\cdot\underline{J}, \overline{I}\cdot\overline{J}\}$$
$$I \vee J = (\max\{\underline{I}, \underline{J}\}, \max\{\overline{I}, \overline{J}\}) \qquad I \wedge J = (\min\{\underline{I}, \underline{J}\}, \min\{\overline{I}, \overline{J}\})$$
$$I <_{\mathbf{I}\mathbb{Q}} J \text{ iff } \overline{I} < \underline{J}$$

Fig. 1. Selected operations on the rational intervals.

Definition 14. *The **interval domain** is defined to be the rounded ideal completion* $\mathsf{RIdl}(\mathbf{I}\mathbb{Q}, \supsetneq)$ *of open rational intervals ordered by reverse strict inclusion. We denote the interval domain by* $(\mathbf{I}\mathbb{R}, \subseteq)$*, and call elements of* $\mathbf{I}\mathbb{R}$ ***interval reals**.*

Note that we can recover the real numbers from $\mathbf{I}\mathbb{R}$. This is because the real numbers can be identified with intervals consisting of only one element. Those intervals are the largest with respect to reverse inclusion. As such, we recover the real numbers by looking at the strongly maximal elements.

Definition 15. *The set* $\mathbb{R}$ *of **real numbers** is defined to be the collection of strongly maximal elements of* $\mathbf{I}\mathbb{R}$*.*

Note that from Definition 15 we directly obtain the apartness relation on the real numbers. This is because every DCPO has an apartness relation on its strongly maximal elements that is irreflexive, symmetric, cotransitive, and tight by Proposition 12.

4 Strong Maximality and Locatedness

To construct a real number using Definition 15, we need to do two things. We must describe an interval real (i.e. an element of $\mathbf{I}\mathbb{R}$), which describes all approximations to the real number, and then we must show this interval real is strongly maximal. Giving a direct proof for such facts is rather complicated, and for that reason, we give a characterization for strong maximality of interval reals in this section.

The characterization of strong maximality that we use, is based on *locatedness*. Our notion of locatedness is similar to the locatedness condition of Dedekind cuts, but phrased using intervals.

Definition 16. *An interval real* $x \in \mathbf{I}\mathbb{R}$ *is **order located** (or simply **located**) if for any rational interval* I *there exists a rational interval* $J \in x$ *such that either* $\underline{I} < \underline{J}$ *or* $\overline{J} < \overline{I}$*.*

Before we show that order locatedness coincides with strong maximality, we will need the following characterization of Hausdorff separatedness.

Lemma 17. *Interval reals* x *and* y *are Hausdorff separated iff there are non-intersecting rational intervals* I, J *such that* $I \in x$ *and* $J \in y$*.*

Theorem 18. *An interval real x is order located if and only if x is strongly maximal.*

Proof. Suppose that x is order located and let I, J be intervals such that $\downarrow I \ll \downarrow J$ (which, by Lemma 22 is equivalent to $I \supsetneq J$), and we are to decide if $\downarrow I \ll x$ (i.e. $I \in x$) or $\downarrow J$ and x are Hausdorff-separated.

Since $I \supsetneq J$, we can consider an interval $L = (\underline{I}, \underline{J})$. By order locatedness, there exists an interval $K \in x$ such that either $\underline{L} = \underline{I} < \underline{K}$ or $\overline{K} < \overline{L} = \underline{J}$. In the latter case we know that K and J are completely disjoint: by Lemma 17 x and J are Hausdorff-separated. In the former case, we consider an interval $R = (\overline{J}, \overline{I})$, and locate it within x. We get an interval $K' \in x$ such that either $\underline{R} = \overline{J} < \underline{K'}$ or $\overline{K'} < \overline{R} = \overline{I}$; we again consider two cases. In the latter case we have $\overline{K'} < \overline{I}$ and $\underline{I} < \underline{K}$. Since x is a rounded ideal, there exists an interval $N \in x$ which refines both K and K': $K, K' \supsetneq N$. It follows that $I \supsetneq N$, and, therefore $I \in x$.

In the former case we have $\overline{J} < \underline{K'}$. Then J and K' are disjoint, and it follows that x and $\downarrow J$ are Hausdorff-separated by Lemma 17.

For the converse, we suppose that I is an interval, and we need to locate it within x. We can always find a smaller interval $I \supsetneq J$; so $\downarrow I \ll \downarrow J$. By strong maximality, either $\downarrow I \ll x$ (equivalently, $I \in x$), or x and $\downarrow J$ are Hausdorff separated.

In the former case, by roundedness of x, there exists some interval $K \in x$ such that $I \supsetneq K$, and it follows that $\underline{I} < \underline{K}$.

In the latter case, by Lemma 17 there are $J_1 \subsetneq J$ and $J_2 \in x$ such that J_1 and J_2 do not intersect. If J_1 lies to the left of J_2, then $\underline{I} < \overline{J} < \overline{J_1} \leq \underline{J_2}$. If J_1 lies to the right of J_2, then $\overline{J_2} \leq \underline{J_1} < \overline{J} < \overline{I}$.

From Theorem 18, we directly obtain that our real numbers are equivalent to the Dedekind real numbers, which was originally proven by De Jong in [16].

Corollary 19 ([16, Theorem 102]). *The set of real numbers from Definition 15 is equivalent to the set of Dedekind real numbers.*

As an application of Theorem 18, we construct the inclusion from the rational numbers to the real numbers. First, we show that every rational number gives rise to an interval real.

Definition 20. *Let q be a rational number. We define an interval real $\lceil q \rceil$ to be the supremum: $\bigcup\{\downarrow I \mid q \in I\}$.*

Note that the supremum exists in Definition 20, because the set $\{\downarrow I \mid q \in I\}$ is directed. Intuitively, this definition says that approximations of $\lceil q \rceil$ are given by intervals that contain q.

Before we show that $\lceil q \rceil$ actually gives rise to a real number, we characterize the way below relation for $\lceil q \rceil$.

Lemma 21. *For all $q \in \mathbb{Q}$ and rational intervals I, we have $\downarrow I \ll \lceil q \rceil$ iff $q \in I$.*

Proof. Follows from Lemma 5 and the definition of $\lceil - \rceil$.

If $x \in \mathbf{I}\mathbb{R}$, then from Lemma 5 we know that $I \in x$ implies $\downarrow I \ll x$, as in any continuous DCPO. However, for $\mathbf{I}\mathbb{R}$ we also have the converse, giving us a characterization of the approximation of interval reals by rational intervals.

Lemma 22. *For any $I \in \mathbf{I}\mathbb{Q}$ and any $x \in \mathbf{I}\mathbb{R}$, we have $\downarrow I \ll x$ iff $I \in x$.*

Proof. Follows from the density of rationals; see also [17, Lemma 100].

Now we show that $\lceil q \rceil$ actually is strongly maximal.

Proposition 23. *For any $q \in \mathbb{Q}$, the interval real $\lceil q \rceil$ is strongly maximal.*

Proof. We use Theorem 18 and show that $\lceil q \rceil$ is order located. Given an interval $I \in \mathbf{I}\mathbb{R}$, we know that either $\underline{I} < q$ or $q < \overline{I}$, by cotransitivity of rational numbers.

Without loss of generality, suppose that $\underline{I} < q$. Then we consider an interval $J = (\frac{\underline{I}+q}{2}, q+1)$. By Lemmas 21 and 22, we have that $J \in \lceil x \rceil$. Furthermore, $\underline{I} < \underline{J} = \frac{\underline{I}+q}{2}$, concluding the proof.

5 Real Arithmetic

So far we have defined the set $\mathbb{R}$ of real numbers and the inclusion $\mathbb{Q} \to \mathbb{R}$. In this section we define the arithmetic operations (addition, multiplication, subtraction, and division) on $\mathbb{R}$, the order $<$, and the lattice operations. More specifically, we show that $\mathbb{R}$ forms an Archimedean field.

Definition 24 ([35, Definition 11.2.7]). *An **ordered field** consists of a set F together with elements $0 \in F$ and $1 \in F$, binary operations $+, *, \min, \max : F \to F \to F$, binary relations $\leq, <, \#$, and operations $- : F \to F$ and $(-)^{-1} : \{x \in F \mid x \# 0\} \to F$. The operations need to satisfy a number of standard laws [35, Definition 11.2.7].*

*A field F is called **Archimedean** if for all $x, y \in F$ such that $x < y$, there merely exists a $q \in \mathbb{Q}$ such that $x < q < y$.*

To define these operations on $\mathbb{R}$, we first lift a corresponding operation on intervals from Figure 1 to the level of interval reals. After that, we show that the lifted operation maps real numbers to real numbers, i.e. they preserve strongly maximal elements. To prove that these operations preserve strong maximality, we use ideas from Bauer and Taylor [5]. More specifically, we define alternative notions of *locatedness* of interval reals, namely *arithmetic locatedness* and *multiplicative locatedness*, that are used to prove locatedness. We define all the operations in the same three steps: 1. identify the corresponding operation rational intervals; 2. lift the operation to interval reals; 3. prove that the lifted operation preserves strong maximality. To lift the functions to real numbers, we use the extension from Definition 6. We define a monotone function $f : \mathbf{I}\mathbb{Q}^n \to \mathbf{I}\mathbb{Q}$, and we define its extension $f_* : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}$ as follows

$$f_*(x_1, \ldots, x_n) = \bigcup \{\downarrow (f(I_1, \ldots, I_n)) \mid I_1, \ldots, I_n \in x_1, \ldots x_n\}.$$

We use the following properties of the lifting of operations.

Lemma 25. *The following statements hold.*

- $f_*(\downarrow I_1, \ldots, \downarrow I_n) \subseteq \downarrow f(I_1, \ldots, I_n)$;
- If $I_1 \in x_1, \ldots, I_n \in x_n$ then $f(I_1, \ldots, I_n) \in f_*(x_1, \ldots, x_n)$;
- If $K \in f_*(x_1, \ldots, x_n)$ then there are $I_1 \in x_1, \ldots, I_n \in x_n$ such that $f(I_1, \ldots, I_n) \subsetneq K$.

We now show how to use this lifting to define operations on interval reals, and show that they preserve strong maximality.

Additive Inverse. We start by defining the additive inverse function $-x$. It is a lifted extension of the corresponding function on the rational intervals:

$$-x = \bigcup\{\downarrow -I \mid I \in X\}.$$

Lemma 26. *If $x \in \mathbf{I}\mathbb{R}$ is order located, then so is $-x$.*

Proof. Suppose that $I \in \mathbf{I}\mathbb{Q}$. We are to locate I within $-x$. First, we use order-locatedness of x w.r.t. $-I$: there exists $J \in x$ such that $\underline{-I} < \underline{J} \vee \overline{J} < \overline{-I}$. Then, by Lemma 25 $-J \in -x$, and, furthermore, $\underline{I} < \underline{-J} = -\overline{J}$ or, similarly, $\overline{-J} < \overline{I}$.

Addition. Addition on reals is defined as follows

$$x + y = \bigcup\{\downarrow (I + J) \mid I \in x, J \in y\}.$$

In order to show that addition preserves strong maximality, we use the auxiliary notion of *arithmetic locatedness*.

Definition 27. *An interval real $x \in \mathbf{I}\mathbb{R}$ is **arithmetically located** if for any rational number $q > 0$ there is a rational interval $I \in x$ such that the interval size $|I|$ satisfies $|I| < q$.*

Proposition 28. *Let x be arithmetically located, and y be order located. Then $x + y$ is order located.*

Proof. Let I be an interval that we are to locate in $x+y$. By arithmetic locatedness, there exists a $J_0 \in x$ with $|J_0| < |I|$. Writing it out, we have $\overline{J_0} - \underline{J_0} < \overline{I} - \underline{I}$, or, equivalently, $\underline{I} - \underline{J_0} < \overline{I} - \overline{J_0}$.

We then use order locatedness of y with respect to the interval $(\underline{I} - \underline{J_0}, \overline{I} - \overline{J_0})$. We get an interval $J_1 \in y$ such that either $\underline{I} - \underline{J_0} < \underline{J_1}$ or $\overline{J_1} < \overline{I} - \overline{J_0}$. Equivalently, $\underline{I} < \underline{J_0 + J_1}$ or $\overline{J_0 + J_1} < \overline{I}$. And, furthermore, by Lemma 25 we have $J_0 + J_1 \in x + y$, thus locating I in $x + y$.

If we show that every order located interval real is arithmetically located, that would conclude the construction of the addition operation on reals. In order to show this, we use the following auxiliary lemma.

Lemma 29. *Let $x \in \mathbf{I}\mathbb{R}$ be order located and let $I \in x$. Suppose that J_0, J_1 are overlapping intervals that cover I. Then there exists $J' \in x$ such that either $J_0 \supsetneq J'$ or $J_1 \supsetneq J'$.*

Proposition 30. *If $x \in \mathbf{I}\mathbb{R}$ is order located, then it is arithmetically located.*

Proof. Suppose that x is order located and $q > 0$ is a rational number. By roundedness, x contains some interval $I \in x$. Then we can cover the whole interval I with n overlapping intervals of size q, for some natural number n. We then use induction on n and Lemma 29 to find some interval in x that is strictly included in one of the covering intervals. By construction, the size of that interval will be strictly smaller than q.

Multiplication. For defining multiplication and proving that it preserves strong maximality, we follow the same approach as for addition. As the operation itself we take the lifting

$$x * y = \bigcup \{\downarrow (I * J) \mid I \in x, J \in y\}.$$

In order to show that this operation preserves strong maximality we use an intermediate notion of multiplicative locatedness:

Definition 31. *An interval real x is **multiplicatively located** if for any rational interval J that lies to the right of 0 (i.e. $0 < \underline{J}$), there exists an interval $K \in x$ such that K lies to the right of 0, and $\underline{J} \cdot \overline{K} < \overline{J} \cdot \underline{K}$. Equivalently, we can say $\overline{K}/\underline{K} < \overline{J}/\underline{J}$.*

Lemma 32. *Suppose that x is an order located interval real that is positive (i.e. there is some $I \in x$ that lies to the right of 0). Then x is multiplicatively located.*

While the lemma above is stated only for positive reals, by playing around with signs and using multiplicative locatedness we can show the following.

Lemma 33. *Suppose that x and y are strongly maximal interval reals. Then $x * y$ is strongly maximal as well.*

Multiplicative inverse. The multiplicative inverse requires special treatment, because it is defined only for interval reals apart from zero. For this reason, we do not define the multiplicative inverse by lifting an operation, but instead, we define it as a particular supremum.

We define a reciprocal of rational intervals as $I^{-1} = (\overline{I}^{-1}, \underline{I}^{-1})$, and we define the reciprocal of interval reals as the following supremum:

$$x^{-1} = \bigcup \{\downarrow (I^{-1}) \mid I \in x \wedge I \# 0\}.$$

This is the supremum over the set $\{\downarrow (I^{-1}) \mid I \in x \wedge I \# 0\}$ ranging over the intervals in x that do not contain 0 (denoted, abusing the notation slightly, as $I \# 0$). This set is always semidirected; furthermore it is inhabited, and therefore directed, whenever x is apart from zero.

Remark 34. The operation I^{-1} is only defined for intervals that do not contain zero – otherwise the operation does not map intervals to intervals. In type theory this is represented by the signature

$$(-,-)^{-1} : (\sum_{I:\mathbf{I}\mathbb{Q}} I\#0) \to \mathbf{I}\mathbb{Q}.$$

The reciprocal not only takes an interval as its argument, but also a proof that it is apart from 0. That means that the directed family in the definition of x^{-1} is represented in type theory as a function with the signature $(\sum_{I\in x} I\#0) \to \mathbf{I}\mathbb{R}$, given by $(I, H) \mapsto \downarrow\big((I, H)^{-1}\big)$.

Similarly, the inverse operation for interval reals takes a proof of apartness from 0 as one of the arguments.

To show that the multiplicative inverse maps real numbers to real numbers, we prove the following:

Lemma 35. *If x is order located and apart from 0, then x^{-1} is also order located.*

Lattice Structure. Finally, we define the lattice operation on the real numbers (minimum, maximum), and the strict order. The strict order is defined as follows: given interval reals x, y, we say that $x < y$ if there are intervals $I \in x, J \in y$ such that $I <_{\mathbf{I}\mathbb{Q}} J$. For the minimum and maximum, we use the same approach as for addition. We show that these these operations preserve strongly maximal elements, using only order locatedness.

6 Arithmetic Laws

Finally, we prove that the real numbers $\mathbb{R}$, together with the operations defined in the previous section, form a constructive ordered field. Some of the constructive field laws follow automatically: intrinsic apartness is reflected by Scott-continuous functions by Proposition 8. Thus, addition and multiplication automatically reflect apartness, as they are defined as Scott continuous extensions of operations on rational intervals. For the remaining constructive field laws, we need to put in a bit more work. Due to space reasons, we only sketch the proof for associativity of addition, and we note that that similar ideas are used to prove the other laws.

Proposition 36. *For any $x, y, z \in \mathbb{R}$ we have $(x + y) + z = x + (y + z)$.*

Proof. First of all, we notice that since we are working with strongly maximal elements, in order to show an equality it suffices to find a common upper bound h such that $(x + y) + z \subseteq h$ and $x + (y + z) \subseteq h$. Secondly, since we are working with an expression with three variables x, y, z, we introduce an intermediate *ternary* version of addition: we write $h(x, y, z)$ for an "unbiased" addition that sums the three numbers together directly

$$h(x, y, z) = \bigcup\{\downarrow(I_1 + I_2 + I_3) \mid I_1 \in x, I_2 \in y, I_3 \in z\}.$$

This approach of using "unbiased" operations is inspired by the "unbiased" monoidal products in monoidal categories [29, Section 3.1]; a similar trick was used to show associativity of smash products in the context of homotopy type theory [30].

Let us then look at how to show $(x+y)+z \subseteq h(x,y,z)$ for all x,y,z. Since h is defined as an extension of a monotone function on the basis, by the universal property it suffices to show $(\downarrow I_1 + \downarrow I_2) + \downarrow I_3 \subseteq \downarrow(I_1 + I_2 + I_3)$. Note that the addition symbol on the right hand side represents addition of rational intervals, not addition of reals. This inclusion then holds by Lemma 25.

Theorem 37. *$\mathbb{R}$ is an Archimedean constructive ordered field.*

7 Conclusions and Related Work

In this paper we presented a formalization of real numbers in the setting of univalent mathematics. We constructed the set of real numbers and we showed that it is an ordered archimedean field. In our formalization we took a novel approach to formulating Dedekind reals using domain theory, and several results were proven more generally. In the future work we would like to show completeness of $\mathbb{R}$ as well.

The topic of constructive real numbers and its formalization has received a lot of attention and has been studied broadly, going back all the way to the pioneering work of Bishop [7,8]. For an overview, see, for example, the surveys [23] (with a focus on constructivity, type theory and domain theory) and [9] (with a focus on formalization). We finish our paper by discussing selected related work on interval reals and domain theory in univalent foundations, and on constructive formalizations of real numbers in type theory.

Domain theory and interval reals. The domain $\mathbf{I}\mathbb{R}$ of interval real numbers was used to study computations with real numbers in the setting of domain theory, for example in semantics of RealPCF [20,19,21]. The interval domain has also been studied in the context of realizability [4], for the purposes of extracting programs for computing with exact real numbers.

In terms of formalizations of domain theory in type theory, we build upon the recent work of De Jong and Escardo on domain theory in the context of univalent mathematics [16,15,18,27].

Finally, while not directly related to interval reals, we would like to mention the work of Bauer and Taylor on constructing Dedekind reals in the context of Abstract Stone Duality [5]. Their work has also served as an inspiration to ours, especially with regard to different notions of locatedness.

Formalization of reals in type theory. Specifically in the context of homotopy type theory/univalent foundations, real numbers have already been considered (with both Dedekind and Cauchy flavors) in the HoTT book [35, Chapter 11], but were not formalized at the time. Dedekind reals were later formalized as part of the UniMath library by Catherine Lelay. The formalization of Cauchy reals usually requires use of quotients and/or countable choice. However, the approach in the HoTT book sidesteps those issues by using a higher inductive-inductive types. This approach has been extended to generic metric space completion and formalized in [25], and it was further used for formalizing synthetic topology [6]. Another approach to the real numbers in univalent foundations is given by the Escardó-Simpson reals [20], which are equivalent to the Cauchy reals [11]. Those have been formalized by Ghica and Ambridge [24,2]. Booij also studied *locators* in univalent foundations [10]. Locators are an additional structure on top of a constructive field, such as the Dedekind reals, and it allows one to assign a decimal expansion to a Dedekind real. Booij also showed that a Dedekind real is a Cauchy real if and only if there is a locator for that number.

It is also worth mentioning the formalization projects from Nijmegen related to real numbers. Initially, as part of the FTA project [13] (formalization of the fundamental theorem of algebra), the real numbers were defined axiomatically, as an abstract interface, to facilitate proof modularity. Later, Niqui and Geuvers [22] developed an implementation of that interface, based on Cauchy reals. This implementation became the basis for FTA, and later became a part of C-CoRN [14], but the implementation was not suitable for extraction and evaluation. O'Conor then proposed another formalization of Cauchy reals [32,33] aimed at extracting and running programs. This approach was further refined by Krebbers and Spitters [28], utilizing the type class approach for formalizing algebraic hierarchies.

Among other formalizations, there is the ALEA Coq library [3], which builds up monadic semantics for a probabilistic programming language based on the real interval. However, the real interval is axiomatized as an abstract type and is not implemented. Another formalization of reals in Coq [12] defines real numbers through (coinductive) infinite streams. The formalization of real numbers in LEGO [26] is similar to ours, as it is based on (converging) nested rational intervals, but the general setting is quite different, and they forgo domain theory.

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