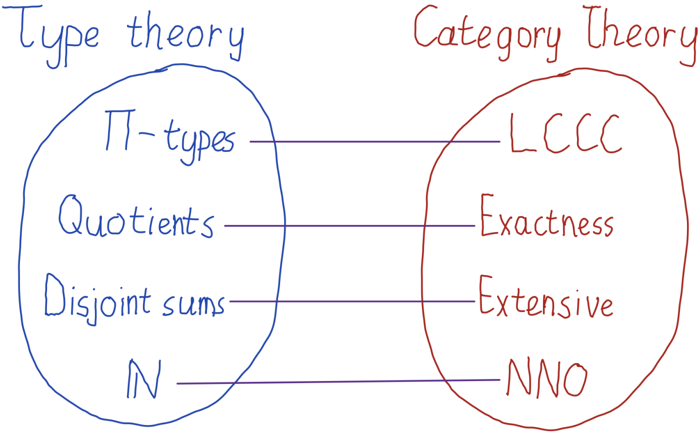
The Internal Language of Univalent Categories

Niels van der Weide

LICS25, June 23, 2025

Link to slides: https://nmvdw.github.io/pubs/lics2025.pdf

Type Theory and Category Theory



Internal Language Theorems

Theorem (Theorem 6.1 in Clairambault&Dybjer 2014[1])

We have a biequivalence betweeen the bicategories

- $\mathsf{CwF}_{\mathsf{dem}}^{\sum,=_{\mathsf{ext}}}$: *democratic comprehension categories with extensional identity types and sigma types*
- FinLim: *finitely complete categories*

This biequivalence can be extended to $\prod$-types and LCCCs

Note: this corrects a result by Seely[2] using ideas by Hofmann[3]

[1] Clairambault, Pierre, and Peter Dybjer. The biequivalence of locally cartesian closed categories and Martin-Löf type theories.

[2] Seely, Robert. Locally Cartesian closed categories and type theory

[3] Hofmann, Martin. On the interpretation of type theory in locally cartesian closed categories

Internal Language Up To Isomorphism

Final sentence of the paper by Clairambault and Dybjer:

So we can ask whether Martin-Löf type theory with extensional identity types, $\sum$- and $\prod$-types is an internal language for lcccs? And we can answer, yes, it is an internal language 'up to isomorphism'.

Internal Language **Up To Isomorphism**

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Univalent Foundations

- **Univalent foundations**[4] is a version of Martin-Löf Type Theory extended with the **univalence axiom**
- **Univalence axiom**: identity of types is the same as equivalence

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- **Univalence axiom**: identity of types is the same as equivalence
- Univalent foundations allows for a **more refined view of identity of structures**: structures are identified up to equivalence
- For instance: groups/ring are identified up to isomorphism

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- Univalent foundations allows for a **more refined view of identity of structures**: structures are identified up to equivalence
- For instance: groups/ring are identified up to isomorphism
- Properties are automatically invariant under equivalence

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Category Theory in Univalent Foundations

- In **univalent foundations**, there are two notions of category: **univalent** categories and **strict** categories
- **Strict categories**: categories indentified up to isomorphism
- **Univalent categories**[5]: categories identified up to adjoint equivalence

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We can thus consider **internal language theorems** for both notions of category, and we consider **internal language theorems for univalent categories**

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Main Theorem

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We have a biequivalence betweeen the bicategories

- DFLCompCat*: univalent democratic comprehension categories that support unit types, equalizer types, binary product types, and strong* $\sum$*-types*
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The proof is formalized using UniMath[6]

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This Talk

There are many interesting aspects to the proof

- **Necessity of non-discrete categorical models**: we use comprehension categories instead of CwFs
- **Transporting properties/structure along equivalences** is free if we assume univalence
- **Displayed bicategories/biequivalences**[7]: modularly construct of bicategories and biequivalences
- **Local properties**[8] generalize a wide variety of type formers

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Discrete versus non-discrete categorical models

There are various notions of categorical models for type theory.
There are **discrete** models that are based on **presheaves**.

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- Natural models[10]

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We use **non-discrete models**.
Note: discrete models are suitable for strict categories[13]

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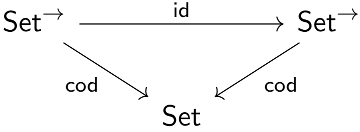
Reject Discreteness! But why?

Recall: a set is a type whose identity type is trivial (i.e., X is a set if for all $x, y : X$ and $p, q : x = y$, we have $p = q$)
The set model can be described as follows



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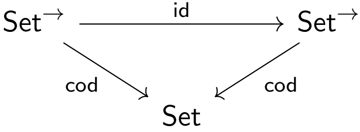

However

- Types in the empty context are the same as sets
- Due to the univalence axiom, the type of sets is not a set itself
- This is because identity of sets corresponds to isomorphism, and there can be nontrivial automorphisms on sets

So: comprehension categories are more suitable than CwFs

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By equivalence induction, we can assume that e is the identity
We use this to:

- transport properties/structure along equivalences
- characterize adjoint equivalences

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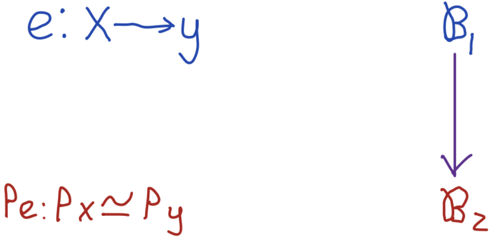
Characterizations of adjoint equivalences:

- to prove that pointwise pseudonatural adjoint equivalences are adjoint equivalences
- to characterize adjoint equivalences of comprehension categories

Characterizing Adjoint Equivalences

Often we want to show that some pseudofunctor reflects adjoint equivalences



Example: underlying pseudofunctor from comprehension categories to fibrations

Characterizing Adjoint Equivalences

If we use **displayed bicategories**, we can use equivalence induction

$\bar{e} : \bar{x} \to \bar{y}$ $\qquad\qquad$ $\mathcal{D}$

$e : x \simeq y$ $\qquad\qquad$ $\mathcal{B}$

By induction on e: we only have to consider morphisms over identities

Conclusion

- Univalent foundations offers an interesting perspective on the categorical semantics of type theory
- We discussed a version of Clairambault's and Dybjer's theorem for univalent categories, and we extended it to various classes uf toposes
- There are many interesting aspects to this proof, and we discussed the usage of non-discrete models and transporting along equivalences
- Preprint: https://arxiv.org/abs/2411.06636v3
- Longer version of this talk: https://www.youtube.com/watch?v=Yk09KrBNUt4