Displayed Monoidal Categories for the Semantics of Linear Logic

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Linear Logic

Categorical Semantics of Linear Logic

Displayed Categories

Displayed Monoidal Categories

What is Linear Logic?

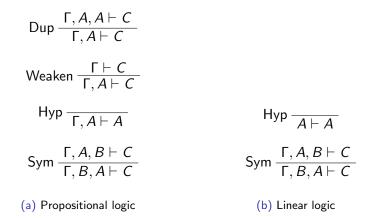
- Linear logic is "the logic of resources"
- Key feature of linear logic: assumptions are used exactly once
- Used for many applications (e.g., quantum physics, separation logic, domain theory)



- To describe linear logic, we give connectives and derivation rules
- Among the derivation rules, there are structural rules, introduction rules, and elimination rules
- Note: often sequent calculus is used for linear logic
- This talk: natural deduction (following a note by Pfenning¹ and the linear λ-calculus by Benton and Wadler)

¹https://www.cs.cmu.edu/~fp/courses/15816-f01/handouts/lnd.pdf

Structural Rules



Note the difference between the Hyp rules!

Connectives

We consider a fragment of **intuitionistic linear logic**. It has the following connectives:

- ▶ linear conjunction: ⊗
- ▶ linear implication: —
- bang modality: ! (you can duplicate assumptions under a !)

People also consider other connectives for linear logic.

- why not modality: ?
- linear negation
- quantifiers

But we shall ignore them in this talk

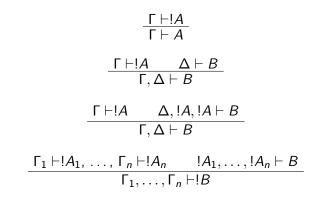
Derivation Rules for \otimes

$$\wedge I \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \\ \otimes E1 \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \qquad \otimes I \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \\ \otimes E2 \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \qquad \qquad \otimes E \frac{\Gamma \vdash A \otimes B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \\ (a) Conjunction \qquad (b) Linear conjunction$$

Derivation Rules for $-\!\!\circ$

$$\rightarrow I \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \qquad - \circ I \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$
$$\rightarrow E \frac{\Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \qquad - \circ E \frac{\Gamma \vdash A \multimap B}{\Gamma, \Delta \vdash B}$$
(a) Implication (b) Linear implication

Derivation Rules for !



We can copy and discard assumptions under the bang modality

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The Semantics of Linear Logic

- To relate the syntax to actual applications, we give denotational semantics
- For example, to relate linear logic to quantum mechanics, we interpret formulas as vector spaces
- Our tool for denotational semantics: category theory

Category Theory and Semantics

Curry-Howard-Lambek correspondence

Logic	Type theory	Category Theory
Formula	Туре	Object
Proof	Term	Morphism
Connective	Type Constructor	Categorical structure

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Note: categorical structure is described via **universal properties** whereas connective/type constructors are described via **introduction and elimination rules**.

Categorical Semantics for Linear Logic

We discussed 3 connectives in our logic:

- ▶ linear conjunction: ⊗
- ▶ linear implication: —
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Categorical Semantics for Linear Logic

We discussed 3 connectives in our logic:

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bang modality: ! (you can duplicate assumptions under a !) Let us start by looking at the semantics of linear conjunction and implication.

Monoidal Categories, what are they?

Basically: Monoidal category = Monoid + category A monoidal category is a category with a multiplication \otimes .

- given objects x, y, we have an object $x \otimes y$
- ▶ given morphisms $f : x \to x'$ and $g : y \to y'$, we have a morphism $f \otimes g : x \otimes y \to x' \otimes y'$

We require \otimes to be associative and unital in a weak sense.

Monoidal Categories and Linear Logic

$$\frac{x: \mathcal{C} \quad y: \mathcal{C}}{x \otimes y: \mathcal{C}} \qquad \qquad \underbrace{A: \operatorname{Prop} \quad B: \operatorname{Prop}}_{A \otimes B: \operatorname{Prop}}$$

$$\frac{f: x \to x' \quad g: y \to y'}{f \otimes g: x \otimes y \to x' \otimes y'} \qquad \qquad \underbrace{\Gamma \vdash A \quad \Delta \vdash B}_{\Gamma, \Delta \vdash A \otimes B}$$
(a) Monoidal Categories (b) Linear Logic

Monoidal Categories and Linear Logic

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(a) Monoidal Categories (b) Linear Logic

For the linear implication, one can use **symmetric monoidal closed categories**.

But what about ! (the bang modality)?

- Giving semantics to the bang modality is more challenging
- There are various options: Lafont categories, Seely-categories, linear categories, linear non-linear models
- This talk: linear non-linear models

Linear-non-linear models: Intuition

- We have a linear world where we cannot duplicate assumptions
- We have a cartesian world where we can duplicate assumptions
- The ! modality jumps from the linear world to the cartesian world and back

Linear-non-linear models: Precisely

A linear-non-linear model is a symmetric monoidal adjunction



where \mathbb{L} is a symmetric monoidal category (\otimes and $-\infty$) and \mathbb{C} is a cartesian category (we can copy and delete hypotheses). We interpret ! as $\mathbb{L} \to \mathbb{C} \to \mathbb{L}$.

Example of a linear-non-linear model

Lifting of complete partial orders gives a model²

$$\omega \operatorname{cpo}_{\perp !} \xrightarrow[\operatorname{lift}]{U} \omega \operatorname{cpo}$$

Here:

- ω cpo: objects are ω cpos, morphisms are continuous maps
- ω cpo₁: objects are pointed ω cpos, morphisms are continuous strict maps
- lift: attaches a minimum element to a ω cpo

²A Mixed Linear and Non-Linear Logic, Benton

Another axample of a linear-non-linear model

A model from abelian groups³

Set
$$\xrightarrow[]{F}$$
 Ab

Here:

- ω Ab: objects are abelian groups, morphisms are homomorphisms
- Set: objects are sets, morphisms are functions
- ► F: free abelian group functor

³A Mixed Linear and Non-Linear Logic, Benton

HOWEVER.....

- There are several methods to give models of linear logic
- Those make use complicated monoidal categories
- The relation model by Lafont uses comonoids
- Other models uses comonads and Eilenberg-Moore categories

Challenge: how do we formalize such monoidal categories in a modular way?

Our paper

- We introduce displayed monoidal categories
- We use them to construct complicated monoidal categories in a modular way
- Nice application of dependent types to category theory
- Formalized using Coq and the UniMath library

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Example of Linear-non-linear models

The relation model of linear logic

$$\mathsf{Comonoid}(\mathsf{Rel}) \xrightarrow[]{U}{} \mathsf{Rel}$$

Here:

- Rel: objects are sets, morphisms are relations
- Comonoid(Rel): objects are comonoids
- The functor C is given by finite multisets (i.e., free comonoid)

Complicated Monoidal Categories: Comonoids

A comonoid (x, ε, δ) in a monoidal category $\mathcal C$ consists of

- ▶ an object *x* : *C*
- ▶ a comultiplication $\varepsilon : x \to x \otimes x$
- ▶ a counit $\delta : x \rightarrow \mathbf{1}$
- Laws: coassociativity and counitality.

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For the tensor, we need to consider comonoids as a whole This does not allow for code reuse (i.e., complicated structures of which comonoids form substructure) We can use the following strategy to define the notion of groups.

- 1. Given a set X, define the type of **group structures** over X
- 2. A group is a set together with a group structure
- This means we define the notion of groups in 2 steps.

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- 1. Given a set X, define the type of **group structures** over X
- 2. A group is a set together with a group structure

This means we define the notion of groups in 2 steps.

Displayed categories formalize this idea for categories

Displayed Categories

A displayed category over a category $\ensuremath{\mathcal{C}}$ consists of

- ► For every object *x* : *C*, a type of structures over *x*
- For all morphisms f : x → y and structures S_x and S_y for x and y respectively, a type of structure-preserving maps

Displayed Categories: Example

The displayed category of groups over sets:

- For every set X, a type of group structures for X
- For all functions $f : X \to Y$ and group structures G_X and G_Y , a type expressing that f is a homomorphism

Building Complicated Structures from Simpler Ones

Displayed categories give **modularity**, because we can **untangle** and **stratify** structures.

Basically: build up complicated structures from simpler structures

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Displayed categories give **modularity**, because we can **untangle** and **stratify** structures.

Basically: build up complicated structures from simpler structures For example:

Product of displayed categories (combines structures)

$$\frac{f: X \to \mathsf{Type}}{h(x) = f(x) \times g(x)}$$

Adding a destructor (i.e. coalgebra structure)

$$f(x) = x \to x^n$$

We can reason about the these parts **independently**, and we can reuse the results in larger proofs.

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Displayed Monoidal Categories, but what are they?

Displayed monoidal categories

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Displayed categories + monoidal categories

Displayed Monoidal Categories, but what are they?

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=

Displayed categories + monoidal categories Note: there also needs to be a suitable interaction between the two concepts Displayed Monoidal Categories, but what are they?

Let S be a displayed category over C.

$$\frac{x:\mathcal{C} \quad y:\mathcal{C}}{x\otimes y:\mathcal{C}} \qquad \qquad \frac{x:\mathcal{C} \quad \overline{x}:S_x \quad y:\mathcal{C} \quad \overline{y}:S_y}{\overline{x}\otimes\overline{y}:S_{x\otimes y}}$$

(a) Monoidal Categories

(b) Displayed Monoidal Categories

Comonoids using displayed monoidal categories

Main idea:

- We define a displayed monoidal category that adds a destructor x → F(x) for a lax monoidal functor F
- \blacktriangleright This way we acquire the counit ε and the comultiplication δ
- We define the full subcategory via a displayed monoidal category, and that gives us the laws
- So: we build up the category of comonoids via smaller pieces and we reason about those smaller parts

Conclusion

- Main take-away: displayed monoidal categories are a technique to modularly build monoidal categories
- In the paper, we define and study displayed monoidal categories
- We apply it to a case study arising from linear logic
- They make the formalization of complicated monoidal categories more convenient and nicer
- Key examples: category of comonoids, Eilenberg-Moore category

Check our paper:

https://dl.acm.org/doi/abs/10.1145/3636501.3636956.