Semantics of two-dimensional type theory

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Higher dimensional type theory

- Martin-Löf's identity type gives types the structure of **higher groupoids**
- This led to the development of **homotopy type theory** (HoTT)
- **Synthetic algebraic topology**: done via HoTT
- **Directed type theory**: directed version of HoTT
- Directed topological spaces are used to study concurrency [1], and directed type theory is conjectured to model such spaces.

[1] Fajstrup, Lisbeth, et al. *Directed algebraic topology and concurrency*. Vol. 138. Berlin: Springer, 2016.

2-Dimensional interpretations of type theory

There are many interpretations of type theory that are 2-dimensional in a certain sense

- The groupoid interpretation by Hofmann and Streicher [2]
- The two-dimensional models by Garner[3]

Interpreted in something like **groupoids**

[2] Hofmann, Martin, and Streicher, Thomas. "The groupoid interpretation of type theory." *Twenty-five years of constructive type theory (Venice, 1995)* 36 (1998): 83-111.

[3] Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

Directed type theory

But directed variants have also been considered

- An interpretation with directed definitional equality[4]
- A syntactical framework for directed type theory[5]
- An interpretation with directed identity types[6]

Interpreted in something like **categories**

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[4] Licata, Daniel R., and Harper, Robert. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

[5] Nuyts, Andreas. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, 2015.

[6] North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

A framework is missing

Problem:

- Garner gave a general notion of 2-dimensional comprehension category, but this only works for **undirected** type theory
- The interpretations of directed type theory are ad hoc

Goal of this talk:

find categorical framework in which one can interpret various flavors of 2-dimensional type theory

The work in this talk has been formalized using UniMath.

Idea

- Use bicategories instead of categories
- Define **comprehension bicategories**.
- For that, we need a bicategorical notion of fibration[7] [8]
- Find suitable instances of comprehension bicategories

[7] Hermida, Claudio. "Some properties of Fib as a fibred 2-category." *Journal of Pure and Applied Algebra* 134.1 (1999): 83-109.

[8] Buckley, Mitchell. "Fibred 2-categories and bicategories." *Journal of Pure and Applied Algebra* 218.6 (2014): 1034-1074.

Comprehension categories

Type theory can be interpreted in **comprehension categories**.

Definition

A **comprehension category** is a *strictly* commuting triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\chi\quad} & \mathcal{C}^{\to} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \mathsf{cod}} & \\ & \mathcal{C} & \end{array}$$

where F is a Grothendieck fibration and where χ preserves cartesian cells.

Fibrations of bicategories

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Global condition:
Given a substitution $s : \Gamma_1 \to \Gamma_2$ and type A in context Γ_2, we get a type $A[s]$ in context Γ_1.
This is *substitution on types*.

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Local condition:
Given a 2-cell $\tau : s_1 \Rightarrow s_2$ where $s_1, s_2 : \Gamma_1 \to \Gamma_2$, and a term $t : A[s_1]$, we get a term of type $A[s_2]$.
(think of 2-cells $\tau : s_1 \Rightarrow s_2$ as reductions from s_1 to s_2)

Comprehension bicategories

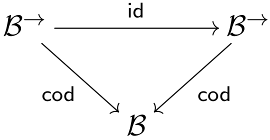
A **comprehension bicategory** is a *strictly* commuting triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad \chi \quad} & \mathcal{B}^{\rightarrow} \\ & {\scriptstyle F}\searrow \qquad \swarrow{\scriptstyle \mathsf{cod}} & \\ & \mathcal{B} & \end{array}$$

where χ preserves cartesian cells and where F is a global fibration and a local opfibration.

Examples of comprehension bicategories

Given a **locally groupoidal** bicategory $\mathcal{B}$ with pullbacks, take



Examples of comprehension bicategories

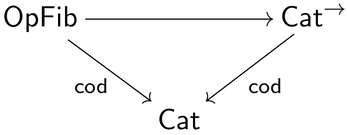
Given a **locally groupoidal** bicategory $\mathcal{B}$ with pullbacks, take

$$\begin{array}{ccc} \mathcal{B}^{\rightarrow} & \xrightarrow{\quad \mathsf{id} \quad} & \mathcal{B}^{\rightarrow} \\ & {\scriptstyle \mathsf{cod}} \searrow \quad \swarrow {\scriptstyle \mathsf{cod}} & \\ & \mathcal{B} & \end{array}$$

This does **not** work for arbitrary bicategories.

Examples of comprehension bicategories

We have the following comprehension bicategory



This can be generalized to arbitrary bicategories by using **internal Street (op)fibrations**.

Conclusion

- We defined a notion of **comprehension bicategory**
- This is a suitable framework in which one can interpret (directed) type theory: we proved **soundness**
- There are general instances of this definition (internal Street fibrations)
- More details can be found in the paper [9].

Further work: look at type formers, completeness

[9] Ahrens, Benedikt, North, Paige Randall, and Weide, Niels van der. "Semantics for two-dimensional type theory." *Accepted to LICS2022*