Enriched Categories in Univalent Foundations

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Univalent Foundations

- **Key aspect of univalent foundations**: the univalent axiom
- **The univalence axiom**: isomorphism of types is the same as equality of types
- The foundations of libraries like UniMath[1].

[1] `https://github.com/UniMath/UniMath`

Category Theory in Univalent Foundations

- In univalent foundations, we are interested in **univalent categories**
- These are categories in which isomorphism between objects is the same as equality between them (*compare to the univalence axiom*)
- Semantically, this is the “right” notion.
- In addition, it is more convenient to work with univalent categories.

Overall Goal

This talk from a broader perspective:

- Develop category theory in univalent foundations
- Formalize it in a proof assistant
- Ultimately: also formalize applications of category theory (i.e., in logic or programming language theory)

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- **Category**: we have objects and between objects, we have a **set** of morphisms
- **Enriched category**: we take the previous definition, but what if we replace set by partial order, abelian group, dcpo, or *an object of an arbitrary monoidal category*?

So: enriched categories are categories whose homsets are endowed with extra structure

Motivation

Applications in mathematics:

- Simplicial homotopy theory [2]
- Strict n-categories can be defined using enriched categories
- Homological algebra [3]

Applications in computer science:

- Domain equations in categories [4]
- Models for the computational λ-calculus [5]
- Models for typed PCF with general recursion [6]
- Enriched effect calculus [7]

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[2] Goerss, Paul G., and John F. Jardine. Simplicial homotopy theory.
[3] Weibel, Charles A. An introduction to homological algebra.
[4] Wand, Mitchell. "Fixed-point constructions in order-enriched categories."
[5] Power, John. "Models for the computational λ-calculus."
[6] Plotkin, Gordon, and John Power. "Adequacy for algebraic effects."
[7] Egger, Jeff, Rasmus Ejlers Møgelberg, and Alex Simpson. "The enriched effect calculus: syntax and semantics."

Goal of This Talk

- As promised by the abstract, I will tell you all something about enriched categories in univalent foundations.
- More specifically, we discuss what a univalent enriched category should be
- The theorems/definitions in this talk are formalized in UniMath[8].

[8] https://github.com/UniMath/UniMath

What are enriched categories?

Suppose that we have a monoidal category $\mathcal{V}$.
Usually, enriched categories consist of the following

- A type O of objects
- For all $x, y : O$ an hom-object $H(x, y) : \mathcal{V}$
- such that we have identity morphisms and compositions

We also require the usual laws to hold (associativity, neutrality)

Enrichments: Why

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- We make use of **enrichments**
- Idea: enriched categories are to groups as enrichments are to group structures.
- This allows for some more reusability.
- In addition, it simplifies some proofs (*structure identity principle*)

Enrichments: Definition

Suppose that we have

- A monoidal category $\mathcal{V}$ with unit $\mathbb{1}$ and tensor $\otimes$

Definition

A $\mathcal{V}$-**enrichment** E of a category C consists of

- a function $\mathsf{E}(-,-) : \mathsf{C} \to \mathsf{C} \to \mathcal{V}$;

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- for $x : \mathsf{C}$ a morphism $\mathsf{Id} : \mathbb{1} \to \mathsf{E}(x, x)$ in $\mathcal{V}$;
- for $x, y, z : \mathsf{C}$ a morphism $\mathsf{Comp} : \mathsf{E}(y, z) \otimes \mathsf{E}(x, y) \to \mathsf{E}(y, z)$ in $\mathcal{V}$;

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- for $x, y, z : \mathsf{C}$ a morphism $\mathsf{Comp} : \mathsf{E}(y, z) \otimes \mathsf{E}(x, y) \to \mathsf{E}(y, z)$ in $\mathcal{V}$;
- functions $\mathsf{FromArr} : \mathsf{C}(x, y) \to \mathcal{V}(\mathbb{1}, \mathsf{E}(x, y))$ and $\mathsf{ToArr} : \mathcal{V}(\mathbb{1}, \mathsf{E}(x, y)) \to \mathsf{C}(x, y)$ for all $x, y : \mathsf{C}$

We require the usual axioms and that FromArr and ToArr are inverses.

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Comments:

- One might wonder: should univalence interact with enrichment?
- For example, for bicategories we have a local and a global univalence condition.
- However, bicategories are instances of **weak enrichments** (over bicategories).
- We look at a stricter notion, namely enrichments over monoidal categories.

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- Such a principle says that equivalent structures are equal
- For enriched categories, this principle basically says: two univalent enriched categories are equal if we have an equivalence between them
- A more precise statement would say that the type of equalities between enriched categories is the same as the type of adjoint equivalences between them
- We formalized this statement.
- Our main method: univalent bicategories

What's included in the formalization so far

- The univalent bicategory of univalent enriched categories
- Limits and colimits in enriched categories
- Enriched monads, and a construction of Eilenberg-Moore objects in the bicategory of enriched categories
- Various examples: self-enriched categories, change of base, the opposite
- Models of the enriched effect calculus and a couple of examples
- Characterization of enrichments over structured sets

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- Various examples: self-enriched categories, change of base, the opposite
- Models of the enriched effect calculus and a couple of examples
- Characterization of enrichments over structured sets (*in the literature, often simplified definitions of enriched categories are used (eg for posets/abelian groups). We define a general notion of structured set and we characterize enrichments over structured sets via a similar simplified definition*)

Conclusion

Main take-aways of this talk:

- Enriched categories are nice and useful
- Univalence for enriched categories: the underlying category is univalent
- We showed: the bicategory of univalent enriched categories is again univalent
- Some interesting peculiarities happen with univalent enriched categories (change of base)