Model Structures on Toposes

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ABSTRACT. We discuss two results: one by Dugger and one by Beke. Dugger's result states that all combinatorial model categories can be written as a Bousfield localization of a simplicial presheaf category. The site of that category gives the generators and the localized maps are the relations, so more intuitively this says that all combinatorial model categories can be built from generators and relations. The second result by Beke gives a general way on how to find model structures on structured sheaves. If all required definitions can be given in a certain logical syntax, then to verify the axioms for all structured sheaves, we only need to check it for sets. This gives an easy way to find the Joyal model structure for simplicial objects in a topos.

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CHAPTER 1

Introduction

Sheaves are one of the main tools in geometry as they allow us to capture local data. For example, using sheaves one can define manifolds, algebraic varieties, schemes and so on. Manifolds are not just topological spaces, but they also have structure which gives the smooth functions. The same thing holds for algebraic variaties, but they have a structure giving the regular functions. These functions can be restricted to a smaller part, and restrictions of smooth functions are smooth. On the other hand, we can glue these functions. If we start with a collection of functions f_α on U_α which agree on the intersections $U_\alpha \cap U_\beta$, then we can give a unique f on $\bigcup_\alpha U_\alpha$ which is f_α on U_α. This precisely says that the extra structure is given by a sheaf, so these objects can be considered as spaces together with a sheaf.

Using such structures we can define more robust operations, and this way we can work more easily with them. As example one could consider the homotopy groups of manifolds, varieties or schemes. The obvious way to define them is by considering them as a topological space for which homotopy groups are defined. However, this does not take the extra structure we have in account, and in practice this does not always give robust techniques. For schemes this method does not give nice results which is why we need different techniques to define homotopy groups of these objects.

One way to do this is by using model structures. These allow us to define homotopy theories on objects other than topological spaces. We start with objects, which can be seen as generalized spaces, and arrows, which represent the continuous maps between them. Normally for homotopy groups we look at how we can map spheres into objects up to homotopy. However, in general we might not have an obvious choice for the sphere in our generalized spaces, so this method does not work. Instead we use more algebraic topology and the structure is given by three classes of maps. Fibrations, cofibrations and weak equivalences are the main tools in algebraic topology, and these can be used to define homotopy groups. To define a model structure, one needs to say which maps are fibrations, cofibrations and weak equivalences, and these need to satisfy some properties. This definition is more abstract and less obvious, but it gives many more examples.

Model structures have had many applications. Quillen introduced model categories to define simplicial homotopy theory for simplicial sets [**Qui67**], and using the general theory he was able to compare simplicial sets and topological spaces. But the applications of model categories reach much further: it can be applied in algebraic geometry, but also in pure algebra. On the one hand, in homological algebra one of the main techniques is working with resolutions and derived functors. With model categories one can describe these constructions in a general setting which is why the study of model categories is sometimes called *homotopical algebra*. On the other hand, recently Morel and Vovoedsky defined the notion of $\mathbb{A}^1$-homotopy theory on schemes [**MV99**]. This allowed them to prove the Milnor and the Bloch-Kato conjectures in algebraic geometry. More recently, this started the development of derived algebraic geometry [**TV05, TV04, Lur04**].

Our first goal is to prove an interesting property of model structures that states that most model categories can be built from generators and relations. This is similar to the structure theorem of finitely generated abelian groups which states that every finitely generated abelian group is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z} \oplus \bigoplus_{i=1}^{m} \mathbb{Z}/p_i\mathbb{Z}$ for some n, m and primes p_i. However, unlike abelian groups this property does not hold for all model categories, but only for those which are combinatorial. Most examples of model categories which occur in practice, are indeed combinatorial, so the requirement is not strict. Also, for model categories it is more difficult to state that they are built with generators and relations, and that requires some set-up. First of all, we need to find 'the free model categories with certain generators'. This basically imitates the presheaf category $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$. The second ingredient is adding relations for which we need a technique called Bousfield localization. Combining these two techniques we can state that every combinatorial model category is built from generators and relations which is precisely Dugger's theorem [**Dug01c, Dug01a**].

Our second goal is about finding a model structure on categories of sheaves, and the formulation requires more set-up. In many examples we are not just looking at sheaves, but rather at structured sheaves. For example, smooth functions can be added, subtracted and multiplied, so the smooth functions on a manifold form a ring. Instead of looking at sheaves, we look at sheaves with a certain structure like sheaves of rings, sheaves of abelian groups or simplicial sheaves. A sheaf of abelian groups is defined similarly as a sheaf of rings, and a simplicial sheaf is a sheaf where the functions on every open subset form a simplicial set. Our goal is to find techniques which can be used to find a model structure on the category of structured sheaves. However, this is quite complicated at first, because sheaves can be defined on all kinds of spaces, even on the complicated ones. To solve this, we want to reduce this problem somehow to sheaves on simpler spaces for which it will be easier.

Let us clarify the approach using the example of simplicial sheaves. If we look at simplicial object in sets, then we see that those are precisely the simplicial sets, in which case we have a model structure. Somehow we would like to transfer this definition from simplicial sets to sheaves. The first thing to notice is that we can write the relevant definitions in such a way that they also make sense for simplicial sheaves. So, we have the required notions at hand, but do they satisfy the right properties? The answer is yes, and the main tool to show this is Boolean localization. However, for it to work, we need some technical requirement on the definitions, because they have to be 'easy' in a certain sense. Using all this we can state and prove the main theorems of [**Bek00, Bek01**].

Now we give an outline of this thesis which is divided in two parts. In each part we study one of the goals. In Chapter 2 we study the basic theory of model categories and locally presentable categories. The notion of 'locally presentable' is crucial in the theory of model categories, because for these categories we are able to find model structures. Next we look at examples of model categories and the more advanced theory in Chapter 3. We start with two techniques to find model structures, namely Quillen's small object argument and transfer. This already gives a wide variety of examples, and it is crucial for the theory as well. Here we also discuss the more advanced notions like Bousfield localization and homotopy colimits. Now we have sufficient techniques for Chapter 4 where we prove that certain model categories can be built from generators and relations. This is the first goal of our thesis, and after that we enter the second part.

In the second part we give a general way to find model structures on categories of structured sheaves. For this we need to start with some topos theory in Chapter 5. Here we discuss the notions of a sheaf and topos, and we look at Boolean localization. Also, we lay the foundations of interpreting logic in toposes which is the main topic of Chapter 6. There we discuss how we can interpret logical sentences in toposes. Another topic of Chapter 6 are sketches which is another logical language. Lastly, we give the precise theorem and prove it in Chapter 7.

Part 1

Abstract Homotopy Theory

CHAPTER 2

Basic Notions

2.1. Model Categories

A valuable tool of abstract homotopy theory is the notion of a *model category*. In general topology homotopy groups are defined just for topological spaces, but not for objects like schemes or simplicial sets. To define homotopy for such objects, we need a structure which generalizes homotopy theory in a certain way. Important notions of homotopy theory are *fibrations*, *cofibrations* and *weak equivalences* which are certain subclasses of the continuous maps. In algebraic topology weak equivalences are maps which induce isomorphisms on all homotopy groups for all possible base points. A weak equivalence does not have to be a homeomorphism, because it might not have an continuous inverse. Fibrations and cofibrations are certain nice maps which give us important computational tools like the Long Exact Sequence. A model category uses fibrations, cofbrations and weak equivalences as the elementary notion rather than spaces or maps from spheres to the given space. In the following definition we use the notation Mor($\mathscr{M}$) for the category of morphisms of $\mathscr{M}$. The objects are arrows $f : X \to Y$ and a morphism from $f : X \to Y$ to $g : X' \to Y'$ is a commutative square of the form

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & Y' \end{array}$$

Now we define

Definition 2.1.1 (Model Category)**.** Let $\mathscr{M}$ be a category with subclasses *Fib*, *Cof* and *We* of arrows. Then $(\mathscr{M}, We, Fib, Cof)$ is a *model category* iff the following axioms are satisfied

- (M1) The category $\mathscr{M}$ has all finite limits and small colimits.
- (M2) Let f and g be composable morphisms. If two of f, g and $g \circ f$ are in *We*, then so is the the third. This is called the *2-out-of-3 property*.
- (M3) The classes *Fib*, *Cof* and *We* are closed under retracts.
- (M4) Suppose that we have the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g]{} & D \end{array}$$

where $i \in Cof$ and $p \in Fib$. If either i or p is in *We*, then there is a lift $C \xrightarrow{h} B$ in the diagram such that $f = h \circ i$ and $g = p \circ h$.

(M5) We have functors $\alpha, \beta, \gamma, \delta : \mathrm{Mor}(\mathscr{M}) \to \mathrm{Mor}(\mathscr{M})$ such that $f = \beta(f) \circ \alpha(f) = \delta(f) \circ \gamma(f)$. These functors must have the property that for every arrow $f : A \to B$, we have $\alpha(f) \in \mathit{Cof}$, $\beta(f) \in \mathit{Fib} \cap \mathit{We}$, $\gamma(f) \in \mathit{Cof} \cap \mathit{We}$, and $\delta(f) \in \mathit{Fib}$.

Maps in *We*, *Fib*, *Cof* are called *weak equivalences*, *fibrations*, *cofibrations* respectively. Also, maps in $\mathit{Fib} \cap \mathit{We}$ are called *trivial fibrations* and maps in $\mathit{Cof} \cap \mathit{We}$ are called *trivial cofibrations*.

Axioms (M1) to (M3) are easy to check in many examples. Also, often it is the case that one of *Fib* and *Cof* is defined as the class of maps which already has one of the desired lifting properties. If we then can prove that the factorization axiom holds as well, then the other lifting axiom will follow, and this will be made more precise in Chapter 3. However, often it is difficult to check whether factorizations can be made, and the main techniques in Chapter 3 will help us showing this property. The fifth axiom is taken from [**Hov07**], and originally in [**Qui67**] the factorizations were not required to be functorial. However, in most examples the factorization is indeed functorial, and it is a nice property for the theory.

Since model categories are finitely complete and cocomplete, they have a terminal object 1 and an initial object 0. Hence, for objects X we have arrows $X \to 1$ and $0 \to X$. Two objects X and Y are called *weakly equivalent* iff there is a zig-zag of weak equivalences between them . This means that we can find objects $Z_1, \dots, Z_n$ with $Z_1 = X$ and $Z_n = Y$, and weak equivalences f_i for $i \in \{1, \dots, n-1\}$ with either $f_i : Z_i \to Z_{i+1}$ or $f_i : Z_{i+1} \to Z_i$. An object is called *fibrant* iff $X \to 1$ is a fibration, and it is called *cofibrant* iff $0 \to X$ is a cofibration . Not all objects have to be fibrant, but they are always weakly equivalent to fibrant objects. This is because we can make the following factorization

$$X \overset{\sim}{\rightarrowtail} X^{\mathrm{Fib}} \twoheadrightarrow 1 .$$

So X is equivalent to X^{Fib} and the map from X^{Fib} to 1 is a fibration, and thus X is equivalent to a fibrant object. Because the factorization is assumed to be functorial, we get a functor $-^{\mathrm{Fib}} : \mathscr{M} \to \mathscr{M}$ which gives a fibrant replacement . For arrows $f : X \to Y$ we get the square

$$\begin{array}{ccccc} X & \overset{\sim}{\rightarrowtail} & X^{\mathrm{Fib}} & \twoheadrightarrow & 1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\mathrm{Fib}}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ Y & \overset{\sim}{\rightarrowtail} & Y^{\mathrm{Fib}} & \twoheadrightarrow & 1 \end{array}$$

Similarly, we can find for each object a cofibrant replacement $-^{\mathrm{Cof}}$.

As is standard in Category Theory, whenever we define a certain class of objects, we should define their arrows as well. The notion of arrow between model categories is the notion of a *Quillen Functor*.

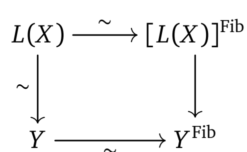
Definition 2.1.2 (Quillen Functor)**.** Let $\mathscr{M}$ and $\mathscr{N}$ be model categories, and let an adjunction $L \dashv R$ be given. Then we call this adjunction a *Quillen adjunction* iff L preserves cofibrations and trivial cofibrations. In this case we call L a *left Quillen functor* or just a *Quillen functor*. The right adjoint R is called a *right Quillen functor*.

A *Quillen equivalence* is defined as a Quillen functor $L \dashv R$ such that for all cofibrant X and fibrant Y a map $L(X) \to Y$ is a weak equivalence iff its transpose $X \to R(Y)$ is a weak equivalence, and two model categories $\mathscr{M}$ and $\mathscr{N}$ are called *Quillen equivalent* iff we have a Quillen equivalence between them. Note that Quillen equivalent model categories might not be equivalent as categories.

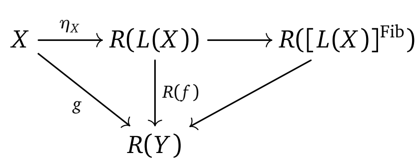
Proposition 2.1.3. *Let $L \dashv R$ be a Quillen adjunction. Then L is a Quillen equivalence iff for all cofibrant X and fibrant Y the composites $X \xrightarrow{\eta_X} R(L(X)) \longrightarrow R([L(X)]^{Fib})$ and $L([R(Y)]^{Cof}) \longrightarrow L(R(Y)) \xrightarrow{\varepsilon_Y} Y$ are weak equivalences.*

PROOF. Let us assume that L is a Quillen equivalence, and let some factorization $L(X) \xrightarrow{i} [L(X)]^{\mathrm{Fib}} \longrightarrow 1$ of $L(X) \to 1$ into a trivial cofibration followed by a fibration be given. The arrow $X \xrightarrow{\eta_X} R(L(X)) \xrightarrow{R(i)} R([L(X)]^{\mathrm{Fib}})$ is the transpose of the arrow i. Since i is a weak equivalence, this arrow is a weak equivalence as well, because L was assumed to be a Quillen equivalence. Factoring $0 \to Y$ into $0 \longrightarrow [R(Y)]^{\mathrm{Cof}} \xrightarrow{p} R(Y)$ into a cofibration followed by a trivial fibration, we can say that $L([R(Y)]^{\mathrm{Cof}}) \xrightarrow{L(p)} L(R(Y)) \xrightarrow{\varepsilon_Y} Y$ is the transpose of the weak equivalence p. Hence, both arrows are weak equivalences.

Next we assume that both the composites $X \xrightarrow{\eta_X} R(L(X)) \longrightarrow R([L(X)]^{\mathrm{Fib}})$ and $L([R(X)]^{\mathrm{Cof}}) \longrightarrow L(R(Y)) \xrightarrow{\varepsilon_Y} Y$ are weak equivalences for cofibrant X and fibrant Y. To show that L is a Quillen equivalence, we first take a weak equivalence $f : L(X) \to Y$ where X is cofibrant and Y is fibrant. Our goal is to show that $g : X \to R(Y)$ is a weak equivalence as well. Next we take a fibrant replacement $[L(X)]^{\mathrm{Fib}}$ and since the factorizations are functorial, we have a map $[L(X)]^{\mathrm{Fib}} \to Y$, because Y is a fibrant replacement of Y. This gives a square



By the 2-out-of-3 property the map $[L(X)]^{\mathrm{Fib}} \to Y$ is a weak equivalence. Now we look at the following diagram



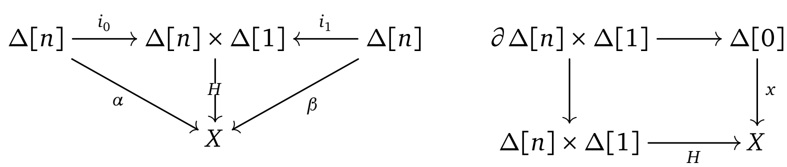
The arrow $X \to R([L(X)]^{\mathrm{Fib}})$ is a weak equivalence by assumption, and the arrow $R([L(X)]^{\mathrm{Fib}}) \to R(Y)$ is a weak equivalence, because Quillen functors preserve weak equivalences between fibrant objects. Hence, $X \to R(Y)$ is a weak equivalence.

To show that $L(X) \to Y$ is a weak equivalence if $X \to R(Y)$ is one, can be done in a similar way. □

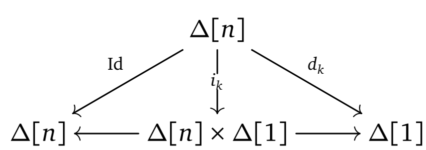
Now we discuss some examples of model categories. At this moment we do not have sufficiently many techniques to prove that they are indeed model categories, but nevertheless it is enlightening. In Chapter 3 we discuss *Quillen's Small Object Argument* which allows us to prove that such structures are indeed model structures. However, in that chapter we will not give the proof for these examples, but rather for two different examples.

Example 2.1.4 (Simplicial Sets)**.** An important example of a model category is given by the category **SSet** of *simplicial sets*. Recall that a simplicial set is defined as a presheaf on Δ^{op}, and that **Top** is the category of topological spaces with continuous maps. In **Top** we have an object Δ^n, called the *standard n-simplex*, which is defined as the subspace of $\mathbb{R}^{n+1}$ consisting of convex combinations of the standard basis vectors. Also, recall that we have a realization functor $|\cdot| : \mathbf{SSet} \to \mathbf{Top}$ which is left adjoint to Sing which sends a topological space X to the simplicial set $X_n = \mathbf{Top}(\Delta^n, X)$. A weak equivalence of simplicial sets is a map $f : X \to Y$ such that $|f| : |X| \to |Y|$ induces an isomorphism on all homotopy groups. Cofibrations are defined to be the monomorphisms, and fibrations are the maps $p : X \to Y$ which have the right lifting property with respect to all horn inclusions $\Lambda^i[n] \to \Delta[n]$. This forms a model structure on **SSet** which can be proven using the techniques of Chapter 3.

Fibrant simplicial sets are called *Kan complexes*, and all simplicial sets are cofibrant. With these notions we can define homotopy groups of a certain class of simplicial sets. The definitions actually work for all simplicial sets, but we need it to be a Kan complex to make it into a group. The n-th homotopy group $\pi_n(X)$ at basepoint $x : \Delta[0] \to X$ of a simplicial set X is defined to be the collection of all maps $\Delta[n] \to X$ mapping $\partial \Delta[n]$ to x up to homotopy where we say that two maps $\alpha, \beta : \Delta[n] \to X$ are homotopic iff there exists a map $H : \Delta[n] \times \Delta[1] \to X$ such that the following two diagrams commute:



We have two degeneracies $d_0, d_1 : \Delta[n] \to \Delta[1]$, and the inclusion i_0 and i_1 are on the first coordinate the identity and on the second coordinate d_0 and d_1 respectively. In a diagram i_k is defined for $k \in \{1, 2\}$ as



One can prove that a map $f : X \to Y$ between Kan complexes is a weak equivalence iff it induces an isomorphism on all homotopy groups.

Simplicial sets are a replacement of topological spaces in abstract homotopy theory. Unlike topological spaces simplicial sets can be described in a combinatorial way, and the category **SSet** is cartesian closed. A simplicial set can capture all the homotopical data of a topological space. To do so, we define the functor Sing sending a topological space X to the simplicial set which is $\mathbf{Top}(\Delta^n, X)$ in degree n. This functor has a left adjoint $|\cdot|$, called the *geometric realization*. If one restricts oneself to the *compactly generated weakly Hausdorff topological spaces*, then one can show these functors give a Quillen equivalence. Hence, simplicial sets generalize topological spaces.

Another important application of simplicial sets is given by the *nerve* functor. Every small category $\mathscr{C}$ gives a simplicial set $\mathscr{N}(\mathscr{C})$ which in degree n consists of all strings of n composable arrows in $\mathscr{C}$. Again the simplicial set contains all the data of the category, and this tells us that simplicial sets also generalize categories. This way we can use certain simplicial sets as model of categories which is a crucial idea in ∞-category theory. Since ∞-categories are not an important topic in this thesis, we will not discuss it any further.

Example 2.1.5 (Chain Complexes)**.** Another interesting model structure is the *projective model structure on chain complexes*. On the category $\mathrm{Ch}_{\geq 0}(R)$ of non negatively graded chain complexes of R-modules we define the weak equivalences to be the maps which induce isomorphisms on all homology groups, and the fibrations are the maps which are surjective in positive degrees. The cofibrations are the maps which are injective and have a projective cokernel. Again to prove this is indeed a model structure requires techniques from Chapter 3 and we will refer the reader for it to [**DS95**].

The cofibrant objects of this model structure are interesting. For a chain complex C the map $0 \to C$ is always injective, and the cokernel at degree n is C_n. Hence, it is cofibrant iff C is projective in every degree. Using the language of model categories we can thus talk about projective resolutions from homological algebra. If N is an R-module, then we can consider the chain complex $K(N,0)$ which is N in degree 0 and 0 in all other degrees. Cofibrant replacements P of $K(N,0)$ thus correspond with projective resolutions of N.

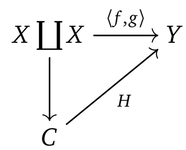
Example 2.1.4 shows up a lot in the theory, and Example 2.1.5 is useful, because it can be used to show that homological algebra can be done using model categories. Basically, model categories give a general way to talk about resolutions, and this generalizes important constructions from homological algebra. Another useful construction in homological algebra is given by *derived functors*. The functor $\mathrm{Tor}(-,B)$, which is the left derived functor of $- \otimes B$, is defined on A by taking a projective resolution P, and then take the homology of $P \otimes B$. This is well-defined up to chain homotopy, and again we can generalize this construction to arbitrary model categories.

However, this requires a small detour on the *homotopy category*. The homotopy category is the localization of the model category with respect to the weak equivalences which means that we added inverses to the weak equivalences in $\mathscr{M}$. This basically copies the definition of localization from commutative algebra. For a ring R and a multiplicative closed subset $S \subseteq R$, the localization R/S satisfies a universal property, namely that for ring homomorphisms $R \to R'$ which send all elements in S to invertible elements, there is a unique extension $R/S \to R'$. We will copy this definition for the homotopy category, and then we say that all functors $F : \mathscr{M} \to \mathscr{C}$ can uniquely be extended to the homotopy category if F sends weak equivalences to isomorphisms.

Definition 2.1.6 (Homotopy Category)**.** Let $\mathscr{M}$ be a model category, and let $\mathrm{Ho}(\mathscr{M})$ be a category with a functor $\gamma : \mathscr{M} \to \mathrm{Ho}(\mathscr{M})$ which send weak equivalences to isomorphisms. Then $\mathrm{Ho}(\mathscr{M})$ is called a *homotopy category* of $\mathscr{M}$ iff for all functors $F : \mathscr{M} \to \mathscr{C}$ which sends weak equivalences to isomorphisms, there must be a unique functor $G : \mathrm{Ho}(\mathscr{M}) \to \mathscr{C}$ such that $G \circ \gamma = F$.

For model categories such homotopy categories can be constructed formally, and for the details on the construction we refer the reader to [**Hov07**]. However, we can construct a equivalent category which is nicer, and for that we imitate the construction of the homotopy category of topological spaces. Here we have a concrete notion of homotopy, and we look at the category whose objects are topological spaces and arrows are homotopy classes of maps. In model categories we can also try to define a notion of homotopy, and then do the same construction.

However, the issue is that we that we have two ways of defining homotopies, namely left and right homotopies. For left homotopies we need cylinder objects of X which are a factorization of the codiagonal map $\nabla : X \coprod X \to X$ into a cofibration followed by a weak equivalence. Note that we can always find cylinder objects C for maps, because every map can be factored as a cofibration followed by a trivial fibration. A left homotopy from $f : X \to Y$ to $g : X \to Y$ is a cylinder object C with a map $H : C \to Y$ such that the following diagram commutes



We can dualize this construction. A path object is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence followed by a fibration, and similarly we get the notion of right homotopy. These notions of homotopy might not coincide for general objects, but they do for objects which are both fibrant and cofibrant, and this gives an equivalence relation. On the subcategory of objects which are both fibrant and cofibrant, we can do the same construction as for topological spaces, so the maps between objects become homotopy classes of maps. This gives a category which we call $\mathscr{M}_{cf}/\sim$. The point is now that $\mathscr{M}_{cf}/\sim$ is equivalent to the homotopy category of $\mathscr{M}$.

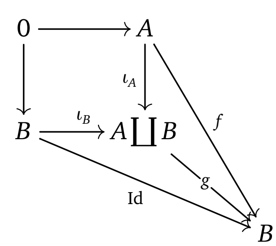
Lastly, we give the notion of a *left derived functor* which generalizes the construction of $\mathrm{Tor}(A, B)$.

Definition 2.1.7 (Left Derived Functor)**.** Let $L : \mathscr{M} \to \mathscr{N}$ be a left Quillen functor. Then the *total left derived functor* $\mathbb{L}L : \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{N})$ of L is defined to be on objects X as $L(X^{\mathrm{Cof}})$ and on arrows as $L(f^{\mathrm{Cof}})$.

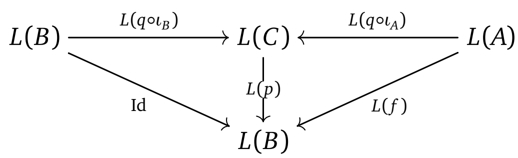
Since the factorization are functorial and assumed to be part of the structure, the definition of $\mathbb{L}L$ makes sense. However, it remains to show this indeed induces a map on the homotopy category, meaning that L sends weak equivalences between cofibrant objects to weak equivalences. This is the case as we show in the following proposition.

Proposition 2.1.8. *Let L be a left Quillen functor, and let X and Y be cofibrant. Suppose, we have a weak equivalence $f : X \to Y$. Then $L(f)$ is a weak equivalence.*

PROOF. Let A and B be cofibrant objects with a weak equivalence $f : A \to B$, and look at the coproduct $A \coprod B$ which is the pushout of of the diagram



Factor $g = p \circ q$ with p a trivial fibration and q a cofibraiton. It is not difficult to prove that the pushout of a cofibration is again a cofibration, so the maps ι_A and ι_B are cofibrations. Since isomorphisms are weak equivalences and f is a weak equivalence, the maps $q \circ \iota_A$ and $q \circ \iota_B$ are weak equivalences by the 2-out-of-3 property, so these maps are trivial cofibrations. We assumed L to be a left Quillen functor, so the map $L(q \circ \iota_A)$ and $L(q \circ \iota_B)$ are weak equivalences. Now we can look at the following diagram



The maps $L(\mathrm{Id}\,)$ and $L(q \circ \iota_B)$ are weak equivalences, so $L(p)$ is a weak equivalence by the 2-out-of-3 property. From the 2-out-of-3 property we can now conclude that $L(f)$ is a weak equivalence, because $L(p)$ and $L(q \circ \iota_A)$ are. □

One can describe the theory as well by looking at *right Quillen functors*. A right Quillen functor is a right adjoint which preserves fibrations and trivial fibrations, and this notion coincides with the one given in Definition 2.1.2. Dually to Proposition 2.1.8 one can prove that these functors preserve weak equivalences between fibrant objects.

2.2. Accessible and Locally Presentable Categories

In this section we discuss two kinds of categories, namely *locally presentable* and *accessible* categories. Later, in Chapter 5, we will see an important example of such categories, namely *toposes*. Their definitions are rather technical and require some setup. Basically, it says that all objects in the category can be constructed from certain *small* objects. Cardinality, or smallness, is not a categorical notion, but it can be described in a categorical way. If we have a set of cardinality λ, and we map it to a disjoint union of more than λ sets, then this map factors through some subset of it which is the disjoint union of at most λ of these sets. This can be generalized in a categorical way, because we can replace the disjoint union by a colimit. In the end of this section we give two characterizations of these notions which are similar to Giraud's theorem, and these are more intuitive.

To discuss the notion of smallness of object, we need to recall the definition of *λ-directed partial orders* where λ is a regular cardinal. If I is a partial order such that every subset of I with cardinality strictly less than λ has an upper bound, then I is called λ-directed. A λ-directed colimit is a colimit over a λ-directed partial order.

Definition 2.2.1 (Smallness)**.** Let $\mathscr{C}$ be a category, and let C be an object of $\mathscr{C}$. For a regular cardinal λ we say that C is *λ-small* iff for every λ-directed partial order I and diagram $F : I \to \mathscr{C}$

$$\mathrm{Hom}(C, \mathrm{colim}_{i\in I} F(i)) \cong \mathrm{colim}_{i\in I} \mathrm{Hom}(C, F(i)).$$

This means that every map from C to a directed colimit factors through some object. To see that this definition generalizes the notion of cardinality of objects, let us discuss some examples. Every finite set $\{x_1, \dots, x_n\}$ is ω-small. If we map it into a colimit $\mathrm{colim}_{j\in I} F(i)$, then every x_i gets mapped into some $F(k_i)$. This gives a finite set of objects $\{F(k_1), \dots F(k_n)\}$, and note that $\{k_1, \dots k_n\}$ has an upper bound k. Hence, the map from $\{x_1, \dots, x_n\}$ to $\mathrm{colim}_{j\in I} F(i)$ factors through $F(k)$, and this gives the desired bijection. More generally, sets of cardinality λ are λ-small which can be proven by exactly the same argument. Another example of small objects are the representable functors. Since $y_C(C')$ is defined as $\mathrm{Hom}(C', C)$ and because the Hom functor is cocontinuous, it commutes with the required colimits, and thus y_C is ω-small.

Note that countable sets are not ω-small. For example, if we have the diagram $i \mapsto \{0, \dots, i\}$ from the partial order ω into **Sets**, then the colimit is ω. However, the identity map on ω does not factor through some $\{0, \dots, i\}$, and thus ω is not ω-small. From this we can conclude that it does not hold that whenever $\lambda' < \lambda$ and C is λ-small, then C must be λ'-small as well. In contrast the opposite does hold: if $\lambda < \lambda'$ and C is λ-small, then C is λ'-small. Countable sets are for example ω_1-small, and we can use precisely the same argument as before to show this. If $\lambda' > \lambda$, then every λ'-directed partial order is λ-directed as well, because the condition of being λ'-directed is stronger. Given a λ-small object C and a diagram F over a λ'-directed partial order, then $\mathrm{Hom}(C, \mathrm{colim}_{i \in I} F(i)) \cong \mathrm{colim}_{i \in I} \mathrm{Hom}(C, F(i))$ holds, because the diagram is λ-directed as well. The converse does not hold, because if $\lambda < \lambda'$ and I is λ-directed, then it might not be λ'-directed.

Next we show that a λ-small colimit of λ-small objects is again a λ-small object. Let I be λ-small and let J be λ-directed partial orders, and suppose we have diagrams $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$ such that every $F(i)$ is λ-small. Then we have the following chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}(\mathrm{colim}_{i \in I} C(i), \mathrm{colim}_{j \in J} F(j)) &\cong \lim_{i \in I} \mathrm{Hom}(C(i), \mathrm{colim}_{j \in J} F(j)) \\
&\cong \lim_{i \in I} \mathrm{colim}_{j \in J} \mathrm{Hom}(C(i), F(j)) \\
&\cong \mathrm{colim}_{j \in J} \lim_{i \in I} \mathrm{Hom}(C(i), F(j)) \\
&\cong \mathrm{colim}_{j \in J} \mathrm{Hom}(\mathrm{colim}_{i \in I} C(i), F(j))
\end{aligned}$$

By definition of the colimit we have the first isomorphism. Since every $C(i)$ is λ-small, we can pull the colimit out of the Hom functor as well. Next we notice that we can interchange λ-small limits and λ-directed colimits in **Sets**. Lastly, we pull the colimit back in to get the desired result. Now we conclude that λ-small colimits of λ-small objects are indeed λ-small.

The following proposition is useful, and a direct consequence of the previous statement. If we can write C as a colimit of λ-small objects, then we can make the colimit directed in such a way that all objects stay λ-small. The problem is that upper bounds might be missing, but to solve that we add them.

Proposition 2.2.2. *Let λ be a regular cardinal and let $\mathscr{C}$ be a category which has all colimits over sets of size at most λ. If we can write C as a colimit of λ-small objects, then we can write C as a λ-directed colimit of λ-small objects.*

PROOF. By assumption we can write C as $\mathrm{colim}_I F$ where $F(i)$ is λ-small for all i. Define a subcategory $\mathscr{D}$ containing the diagram F and all colimits over sets of size at most λ. Note that $\mathscr{D}$ consists only of λ-small objects and that $\mathscr{D}$ is λ-directed, because we added the upper bounds. Every cocone on $\mathscr{D}$ is one over F as well, because in $\mathscr{D}$ we have more objects and arrows. If we have a cocone D over F, then we get one over $\mathscr{D}$, because we get arrows from the colimits to D by the universal property of the colimit. Hence, the category of cocones over $\mathscr{D}$ and over F are isomorphic, and thus the colimit of $\mathscr{D}$ is C as well. □

Now we have developed some techniques to work with small objects and their colimits, and next we introduce the notion of accessibility. This says that certain small objects generate the category using λ-directed colimits. As a technicality we require that all λ-directed colimits exist.

Definition 2.2.3 (Accessible)**.** Let λ be a regular cardinal. A category $\mathscr{C}$ is called λ-*accessible* iff the following two conditions hold

(1) $\mathscr{C}$ has all λ-directed colimits;
(2) There is a set S of λ-small objects such that every object in $\mathscr{C}$ is a λ-directed colimit of S.

A category is called *accessible* if it is λ-accessible for some λ.

If $\mathscr{C}$ is λ-accessible and $\lambda < \lambda'$, then $\mathscr{C}$ might not be λ'-accessible, because it might not have the required colimits. At the moment it still requires some work to check whether a category is accessible or not, but later we shall see some ways to check it more easily. To get a feeling for this notion we shall do some examples by hand

Example 2.2.4 (Presheaves)**.** The category of presheaves on some small category $\mathscr{C}$ is accessible. Since $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ is cocomplete, the first condition is satisfied. The representables are ω-small, and note that every presheaf F is isomorphic to colim $\mathscr{C}^{\mathrm{op}} \downarrow F$. Using Proposition 2.2.2 we see that every presheaf F is an ω-directed colimit of representables, and thus the category $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ is ω-accessible.

Example 2.2.5 (Abelian Groups)**.** Since Ab has a colimits, the first condition of accessibility holds. From algebra we know that every abelian group can be written as colimit of its finitely generated subgroups, and thus we claim that precisely these groups are the generators. If we map $\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}/p_i\mathbb{Z}$ into $\mathrm{colim}_{i \in I} A_i$, then we look at what happens to every generator x_i. Each of these generator gets mapped to some A_{k_i}, and by taking their upper bound, we see that this map factors uniquely through some A_k. Therefore, finitely presentable abelian groups are indeed ω-small. Now we again apply Proposition 2.2.2 to conclude that Ab is ω-accessible.

Example 2.2.6 (Chain Complexes)**.** Since Ab is locally presentable, it easily follows that $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$ is ω-accessible as well. The generators can be taken as the chain complexes which are finitely generated in one degree and zero in all the other degrees. Using a similar argument we can show that these chain complexes are ω-small, and since every chain complex is the colimit of the described generators, it follows by Proposition 2.2.2 that $\mathrm{Ch}_{\geq 0}\,\mathrm{Ab}$ is ω-accessible.

The next notion we consider is *local presentability*. This is slightly stronger than accessibility, but still we can find many examples of locally presentable categories.

Definition 2.2.7 (Locally Presentable)**.** A category $\mathscr{C}$ is called *locally λ-presentable* iff it is λ-accessible and cocomplete, and it is *locally presentable* iff it is locally λ-presentable for some ordinal λ.

The only difference between locally presentable categories and accessible categories is thus that locally presentable categories must have all colimits instead of λ-filtered colimits for some cardinal number. For this reason we have for all locally λ-presentable categories $\mathscr{C}$ and $\lambda' > \lambda$ that $\mathscr{C}$ is locally λ'-presentable as well. Since presheaf toposes are cocomplete, it follows that they are locally presentable by Example 2.2.4. The category of simplicial sets $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ is therefore locally presentable as well. For the same reasons, the category of abelian groups and chain complexes are locally presentable.

Theorem 2.2.8. *Let λ be a regular cardinal. Then the following two statements are equivalent for categories $\mathscr{C}$*

(1) *$\mathscr{C}$ is locally λ-presentable;*
(2) *$\mathscr{C}$ is equivalent to a full reflective subcategory of **Sets**$^{\mathscr{A}^{\mathrm{op}}}$ which is closed under λ-filtered colimits, for some small category $\mathscr{A}$.*

PROOF. Suppose that $\mathscr{C}$ is equivalent to a full reflective subcategory of $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ closed under λ-filtered colimits. This means that we have an adjunction $a \dashv i$ for which we have $a(i(F)) \cong F$ where $i : \mathscr{C} \to \mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$. First, we show that $\mathscr{C}$ is cocomplete. Let $D : I \to \mathscr{C}$ be a diagram in $\mathscr{C}$, and note that we have an induced diagram $I \xrightarrow{D} \mathscr{C} \xrightarrow{i} \mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$. This diagram has a colimit C, and we claim that the colimit of D is $a(C)$. Since $\mathrm{Hom}(a(C), X)$ is isomorphic to $\mathrm{Hom}(C, i(X))$ by adjunction, $a(C)$ is indeed the colimit of the diagram.

Next we assume that $\mathscr{C}$ is locally λ-presentable, and we write $\mathscr{C}_\lambda$ for the full subcategory of generators. Note that $\mathscr{C}_\lambda$ is small, and that we have a diagram

$$\begin{array}{ccc} \mathscr{C}_\lambda & \longrightarrow & \mathbf{Sets}^{\mathscr{C}_\lambda^{\mathrm{op}}} \\ \big\downarrow & & \\ \mathscr{C} & & \end{array}$$

By Kan extension we get an adjunction $a \dashv i$ where $i : \mathscr{C} \to \mathbf{Sets}^{\mathscr{C}_\lambda^{\mathrm{op}}}$, and now we need to show $a(i(C)) \cong C$. If we write C as $\operatorname{colim} D_i$ where D_i is λ-small, then $i(C) \cong \operatorname{colim} y_{D_i}$. Next we can compute $a(i(C))$ as follows

$$a(i(C)) = \operatorname{colim}_{Y(U) \to i(C)} U = \operatorname{colim}_{U \to C} U \cong C$$

because C is the colimit of λ small objects. □

Next we show that functor categories $\mathscr{C}^{\mathscr{D}}$ are locally presentable if $\mathscr{C}$ is locally presentable and $\mathscr{D}$ is small for which we use this theorem.

Example 2.2.9. If $\mathscr{C}$ is locally presentable and $\mathscr{D}$ is small, then $\mathscr{C}^{\mathscr{D}}$ is locally presentable. By Theorem 2.2.8 $\mathscr{C}$ is equivalent to a full reflective subcategory of $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ which is closed under λ-filtered colimits. This means that we have a functor $i : \mathscr{C} \to \mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ with a left adjoint $L : \mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}} \to \mathscr{C}$ such that i preserves λ-filtered colimits and i is an equivalence. Now we claim that $\mathscr{C}^{\mathscr{D}}$ is equivalent to such a subcategory of $\mathbf{Sets}^{\mathscr{A} \times \mathscr{D}}$. Since we have an embedding from $\mathscr{C}$ to $\mathbf{Sets}^{\mathscr{A}}$, we get a functor $i^{\mathscr{D}}$ from $\mathscr{C}^{\mathscr{D}}$ to $(\mathbf{Sets}^{\mathscr{A}})^{\mathscr{D}} \cong \mathbf{Sets}^{\mathscr{A} \times \mathscr{D}}$. We need to show the that this functor has a left adjoint, that it preserves λ-filtered colimits, and that it is an equivalence.

Because λ-filtered colimits are taken coordinate wise and because $i^{\mathscr{D}}$ is defined as i at every coordinate, it preserves λ-filtered colimits. To check that it is an equivalence, we check that it is full and faithful. Since i is faithful, we again have that $i^{\mathscr{D}}$ is faithful since a natural transformation $(\eta_D)_{D \in \mathscr{D}}$ gets mapped to $(i(\eta_D))_{D \in \mathscr{D}}$ and i induces an injection on Hom-sets. Also, if we have a natural transformation $(\eta_D)_{D \in \mathscr{D}}$ in $(\mathbf{Sets}^{\mathscr{A}})^{\mathscr{D}}$ between two functors $i(F)$ and $i(G)$, then at every D we can find a unique preimage τ_D. Now we need to show that $(\tau_D)_{D \in \mathscr{D}}$ is a natural transformation, and this follows from the assumption that i is faithful. From faithfulness follows that only commutative diagrams get mapped to commutative diagrams, and thus i is indeed full.

To show that $i^{\mathscr{D}}$ has a left adjoint, we consider $L^{\mathscr{D}}$. By definition the counit $\varepsilon : i \circ L \Rightarrow \mathrm{Id}$ and unit $\eta : \mathrm{Id} \Rightarrow L \circ i$ such that $(\varepsilon F) \circ (F\eta) = \mathrm{Id}$ and $(G\varepsilon) \circ (\eta G) = \mathrm{Id}$. Since the assignment $H \mapsto H^{\mathscr{D}}$ is functorial, it maps η to $\eta^{\mathscr{D}}$ and ε to $\varepsilon^{\mathscr{D}}$ such that the required diagrams for $i^{\mathscr{D}}$ and $L^{\mathscr{D}}$ commute. Hence, $L^{\mathscr{D}}$ is left adjoint to $i^{\mathscr{D}}$, and now the statement follows.

From this we can conclude for example that the category of simplicial objects in a locally presentable category is again locally presentable, and this gives a third way of proving that $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ is locally presentable. Now we know some examples of locally presentable and accessible categories. Accessible categories satisfy the so-called 'solution set condition', and this is nice. Recall the Freyd adjoint functor theorem

Theorem 2.2.10 (Freyd Adjoint Functor Theorem)**.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor which preserves all colimits. Then F has a right adjoint iff for every object X there is a set $\mathscr{L}$ consisting of arrows $X \to F(Y)$ such that all arrows $f : X \to F(Y)$ can be factorized as*

$$\begin{array}{ccc} X & \xrightarrow{f} & F(Y) \\ {\scriptstyle f_i}\downarrow & \nearrow{\scriptstyle F(g)} & \\ F(Y_i) & & \end{array}$$

with $f_i \in \mathscr{L}$.

The condition in this theorem is called the *solution set condition*. In this thesis however we are not interested in this condition in general, but rather in a specific instance. Given a category $\mathscr{C}$ with a class $\mathbb{W}$ of morphisms, we have an inclusion functor $\mathbb{W} \to \mathscr{C}$. This also gives a functor $\mathrm{Mor}(\mathbb{W}) \to \mathrm{Mor}(\mathscr{C})$. If this functor satisfies the solution set condition, then it means that for all arrows $m : A \to C$ there is a set W_m for which all commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$$

can be factored as

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle w_m} & & \downarrow{\scriptstyle w} \\ C & \longrightarrow & Y & \longrightarrow & D \end{array}$$

where $w_m \in W_m$. If we say $\mathbb{W}$ satisfies the solution set condition, then we mean that $\mathrm{Mor}\,\mathbb{W} \to \mathrm{Mor}\,\mathscr{C}$ satisfies the solution set condition.

Definition 2.2.11 (Accessible Functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be λ-accessible categories. Then a functor $F : \mathscr{C} \to \mathscr{D}$ is called *λ-accessible* iff its preserves λ-directed colimits. A functor is called *accessible* iff it is λ-accessible for some λ.

An *accessible subcategory* of $\mathscr{C}$ is a subcategory $\mathscr{B}$ such that the inclusion $\mathscr{B} \to \mathscr{C}$ is an accessible functor. Now we discuss some properties of accessible categories and accessible functors which help us to find examples of accessible categories. The first one is a way to check for the solution set condition.

Proposition 2.2.12. *Accessible functors satisfy the solution set condition.*

For accessible functors this gives an easy test to see whether they have an adjoint, namely that it preserves the right limits. From this we can conclude that accessible subcategories satisfy the solution set condition at every object. The other property says that accessible categories are preserved under exponentials.

Proposition 2.2.13. *If $\mathbb{D}$ is a small category and $\mathscr{C}$ is an accessible category, then $\mathscr{C}^{\mathbb{D}}$ is accessible.*

The last property on accessible categories we need, can be used to find more examples of accessible categories. It says that the inverse image of an accessible subcategory under an accessible functor is again accessible.

Proposition 2.2.14. *Let $F : \mathscr{D} \to \mathscr{C}$ be an accessible functor, and let $\mathscr{B}$ be an accessible subcategory of $\mathscr{C}$. Then the full subcategory $F^{-1}(\mathscr{B})$ of $\mathscr{D}$, consisting of the objects X such that $F(X)$ is an object in $\mathscr{B}$, is an accessible subcategory of $\mathscr{D}$.*

We will not give the proof here and instead refer the reader to [**AR94**].

CHAPTER 3

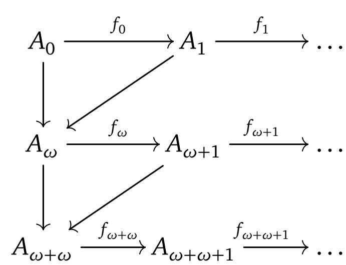
Finding Model Structures

We have defined an abstract notion of model structure, and the question is how to find these in nature. Most axioms of Definition 2.1.1 are easy to check, namely (M1) to (M3). In a lot of examples either the fibrations or the cofibrations are defined in such a way that one of the lifting properties in (M4) trivially holds. However, (M5) is often difficult to check, and hard to do by hand. A general way of constructing such factorizations, is given by *Quillen's small object argument*.

3.1. Quillen's Small Object Argument

Quillen's small object argument constructs the factorization into certain classes of maps if we can find a set which generates these classes. To apply this, we need to find generators for both the cofibrations and the trivial cofibrations. This is often way more convenient than directly constructing the factorizations by hand, and we will discuss some examples where we can apply this method.

Recall that a transfinite composition is defined as the colimit of a diagram over some ordinal. Let α be an ordinal number, and suppose that we have composable maps $f_\beta : A_\beta \to A_{\beta+1}$ for all $\beta < \alpha$ such that for all limit ordinals $\beta < \alpha$ we have $A_\beta \cong \operatorname{colim}_{\gamma<\beta} A_\gamma$. Then we can form the diagram



The colimit of this diagram is defined to be the transfinite composition of all f_β.

Definition 3.1.1. Let $\mathscr{C}$ be a category and let I be a set of morphisms in $\mathscr{C}$. An *I-cellular complex* is a factorization of a map $0 \to X$ as a transfinite composition of pushouts along I. Let Cell(I) be the collection of I-cellular complexes, and note that it is closed under transfinite composition and pushout along I. The *I-cofibrations* are the retracts of the I-cellular complexes, and the *I-injective maps* are the maps which have the right lifting property with respect to I. The collection of I-cofibrations is denoted as Cof(I) and the collection of I-injective maps as Inj(I).

One might wonder why I-cellular complexes are called such, and that is because they resemble the cellular complexes from topology. Take I to be the collection of boundary inclusions $S^{n-1} \to D^n$ and $\emptyset \to D^0$, and let us show that the circle is a I-cellular complexes. To add a point to X, we take the pushout

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ D^0 & \longrightarrow & X^+ \end{array}$$

Note that the map $\emptyset \to D^0$ is in I, and note that in a similar fashion we can add n-cells to X. For the circle we want to add a line and a point, so we start by adding a point to $\emptyset$ to obtain $*$. Next we add a line to $*$ using the following pushout

$$\begin{array}{ccc} S^0 & \longrightarrow & * \\ \downarrow & & \downarrow \\ D^1 & \longrightarrow & S^1 \end{array}$$

The map $S^0 \to *$ is the constant map, and this pushout gives S^1. Now we can factor $\emptyset \to S^1$ as $\emptyset \to * \to S^1$ by composing the pushout maps. If we construct a cellular complex with an infinite amount of cells, we need transfinite composition for infinite ordinals.

Note that whenever f has the right lifting property with respect to I, then it has the right lifting property with respect to every I-cellular complex.

Theorem 3.1.2 (Quillen's Small Object Argument). *Let $\mathscr{C}$ be a locally presentable and locally small category, and let I be a set of maps in $\mathscr{C}$. Then every map $f : A \to B$ can functorially be factorized as $p \circ i$ where $p \in \mathrm{Inj}(I)$ and $i \in \mathrm{Cell}(I)$.*

The first step of the proof is building objects X_α for every ordinal α smaller than some large ordinal λ. Step by step we glue certain things to X, and at every step we have a map $X_\alpha \to Y$. When we arrive at λ, we get the desired factorization. The map $X \to X_\lambda$ is a cellular complex by construction, and to show that X_λ is an I-injective, we use the fact that $\mathscr{C}$ is locally presentable.

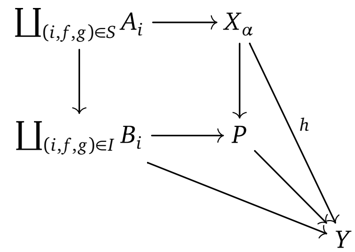
PROOF. Because $\mathscr{C}$ is locally presentable, each domain A_i of an arrow $i \in I$ is μ_i-small for some μ_i, and define λ to be successor the supremum of all μ_i, so $\lambda = \bigcup_i \mu_i$. Since successor cardinals are regular, this means that λ is regular Now we will glue step by step certain things to X such that at every step we have a map to Y, and we will do that using transfinite induction. We start by defining X_0 to be X, and next assume that we have X_α for $\alpha < \lambda$ with a map $h : X_\alpha \to Y$.

To construct $X_{\alpha+1}$ we consider all diagrams of the form

$$\begin{array}{ccc} A_i & \longrightarrow & X_\alpha \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle h} \\ B_i & \longrightarrow & Y \end{array}$$

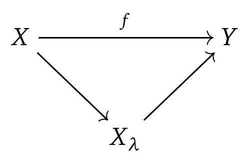
where $i \in I$. Since $\mathscr{C}$ is assumed to be locally small, we have a set S which contains all these diagrams. More concretely, S consists of triples (i, f, g) with $i \in I$, $f : A_i \to X_\alpha$

and $g : B_i \to Y$ such that $g \circ i = h \circ f$. Now we can form the pushout



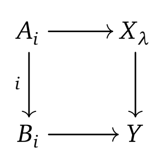
The map $\coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ is given on factor (i, f, g) by the map i, the map $\coprod_{i\in I} A_i \to X_\alpha$ is defined on factor (i, f, g) as the map f, and lastly the map $\coprod_{(i,f,g)\in I} B_i \to Y$ is given on (i, f, g) as g. Since for all (i, f, g) we have $g \circ i = h \circ f$, we indeed have the map from P to Y. For the induction step we thus define $X_{\alpha+1}$ to be P, and we note that now we indeed have a map from $X_{\alpha+1}$ to Y.

For a limit ordinal $\mu \leq \lambda$ we define X_μ as the transfinite composition $\operatorname{colim}_{\alpha<\mu} X_\alpha$. Since at every factor we have a map from X_α to Y and because the required diagrams commute, we get a map $X_\mu \to Y$. With this construction we have obtained an object X_λ and a factorization

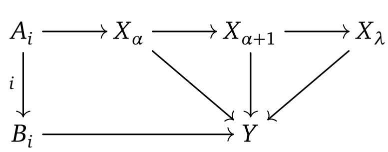


Since the map $X \to X_\lambda$ is defined a transfinite composition of pushouts of I, it is a I-cellular complex. Hence, it remains to show that the map $X_\lambda \to Y$ is an I-injective.

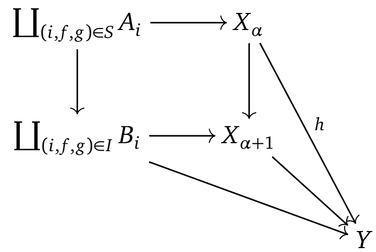
Suppose we have a diagram



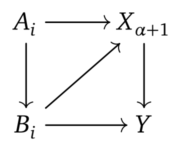
and we need to find a lift. Note that A_i is λ-small from which we can conclude that morphisms from A_i to X_λ factor through some $\alpha < \lambda$. This is the sets $\operatorname{Hom}(A_i, \operatorname{colim}_{\alpha<\lambda} X_\alpha)$ and $\operatorname{colim}_{\alpha<\lambda} \operatorname{Hom}(A_i, X_\alpha)$ are isomorphic, and thus the map $A_i \to X_\lambda$ gives a map $A_i \to X_\alpha$ for some $\alpha < \lambda$. Now we can look at the following diagram



Let us recall for a moment that $X_{\alpha+1}$ was defined as the following pushout



We have the inclusion of A_i and B_i into $\coprod_{(i,f,g)\in S} A_i$ and $\coprod_{(i,f,g)\in I} B_i$ respectively using the upper and lower map in the diagram. This gives us a map from B_i to $X_{\alpha+1}$ which makes the following diagram commute



By composing we can also find the desired lift $B_i \to X_\lambda$ which completes the proof of Quillen's small object argument. □

This theorem is our main tool to find model structures. To find the model structure, we find generating sets for the cofibrations and the trivial cofibrations. Then we apply Quillen's small object argument twice to conclude. If we can construct a model structure this way, then it is called *combinatorial*.

Definition 3.1.3 (Combinatorial Model Category)**.** A model category $\mathscr{M}$ is called *combinatorial* iff it is locally presentable and there is a set I of cofibrations and a set J of trivial cofibrations such that the cofibrations are $\mathrm{Cof}(I)$ and the trivial cofibrations are $\mathrm{Cof}(J)$.

As an application we can already give a way to find combinatorial model structures.

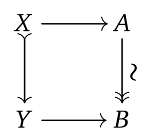
Theorem 3.1.4. *Let $\mathscr{C}$ be a locally presentable category, a subcategory* We *of $\mathscr{C}$ and let two sets I and J of maps be given. Suppose that*

(1) We *satisfies the 2-out-of-3 property and is closed under retracts;*
(2) $\mathrm{Cof}(J) \subseteq \mathrm{Cof}(I) \cap \mathrm{We}$*;*
(3) $\mathrm{Inj}(I) \subseteq \mathrm{We}$.

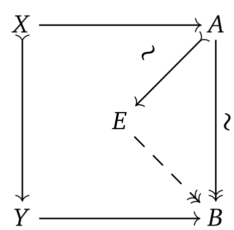
Then we have a combinatorial model structure on $\mathscr{C}$ where the weak equivalences are We*, the cofibrations are* $\mathrm{Cof}(I)$*, and the fibrations are* $\mathrm{Inj}(J)$

Proof. Let us check that the axioms hold, and let us start with the straightforward part. Because $\mathscr{C}$ is locally presentable, it has all small limits and colimits, and thus (M1) holds. By assumption the 2-out-of-3 property is satisfied, and thus (M2) holds as well. Also, we assumed that *We* is closed under retracts and by definition $\mathrm{Cof}(I)$ and $\mathrm{Inj}(\mathrm{Cof}(J) \cap We)$ are closed under retracts, and from this (M3) follows. One of the axioms of (M4) is trivial, because the fibrations are defined to be the maps with the right lifting property with respect to the trivial cofibrations. Using Quillen's small object argument we can factorize maps as a trivial cofibration followed by a map which has the right lifting property with respect to trivial cofibrations. Since we know that the fibrations are precisely the maps which have the right lifting property with respect to trivial cofibrations, we get the desired factorization.

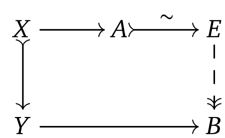
Now we check the other parts of (M5) and (M4). Since $\mathrm{Cof}(J) \subseteq \mathrm{Cof}(I)$ by (2), we have that $\mathrm{Inj}(\mathrm{Cof}(I)) \subseteq \mathrm{Inj}(J)$. Hence, if a map has left lifting property with respect to all cofibrations, then it is a fibration. By Quillen's small object argument we can factorize a map as a cofibration followed by a map which has the right lifting property with respect to the cofibrations, and that map must be a fibration. By (3) this map is a weak equivalence, and thus (M5) holds. Let us denote cofibrations by $X \rightarrowtail Y$, fibrations by $X \twoheadrightarrow Y$, and maps with the right lifting property with respect to cofibrations by $X \dashrightarrow\!\!\!\!\rightarrow Y$. First we show that trivial fibrations have the right lifting property with respect to cofibrations, and for that we start with the following diagram



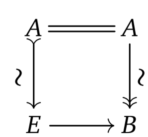
We can factor $A \overset{\sim}{\twoheadrightarrow} B$ as a trivial cofibration followed by a fibration, and by the 2-out-of-3 property the fibration is a weak equivalence as well, and this gives the following diagram



We get a map $Y \longrightarrow E$, because we can find a lift in the following diagram



Also, we have a map from E to A, because we can find a lift in the following diagram



This is because trivial cofibrations have the left lifting property with respect to fibrations. Hence, trivial fibrations have the right lifting property with respect to the cofibrations, and thus (M4) holds. □

Note that the model structure defined in the proof is indeed combinatorial, because the cofibrations and trivial cofibrations are generated by some set.

Example 3.1.5 (Projective Model Structure on Functors)**.** We define a model structure on the category of functors, so let $\mathscr{M}$ be combinatorial category and $\mathbb{D}$ be small. Since we would like to apply Quillen's small object argument, we need that $\mathscr{M}^{\mathbb{D}}$ is locally presentable, and that is why we require that $\mathscr{M}$ is combinatorial. Define a model structure on $\mathscr{M}^{\mathbb{D}}$ where the weak equivalences and the fibrations are defined objectwise, but now the cofibrations are defined to be the maps which have the left lifting property with respect to all trivial fibrations. This model structure is called the *projective model structure* or the *Bousfield-Kan model structure*, and was first described in [**BK87**].

To show that this is indeed a model structure, we use Theorem 3.1.4. Note that $\mathscr{M}^{\mathbb{D}}$ is locally presentable, because $\mathscr{M}$ is locally presentable. Also, (1) obviously holds, because it holds in $\mathscr{M}$ and weak equivalences are defined pointwise. Because $\mathscr{M}$ is combinatorial, there are sets $I_{\mathscr{M}}$ and $J_{\mathscr{M}}$ of generating cofibrations and trivial cofibrations respectively. For an object D of $\mathbb{D}$ and X of $\mathscr{M}$, define a presheaf

$$F_X^D(D') = \coprod_{\alpha \in \mathbb{D}(D,D')} X$$

So, $F_X^D(D')$ has a copy of X for every arrow $f : D \to D'$ in $\mathbb{D}$. We define this presheaf, because a (trivial) cofibration $i : X \to X'$ induces a (trivial) cofibration $\widehat{i} : F_X^D \to F_{X'}^D$, and the generating sets will consists of these maps. Now let $i : X \to X'$ be any cofibration in $\mathscr{M}$, and note that this gives a map $\widehat{i} : F_X^D \to F_{X'}^D$.

Let us check that $\widehat{i}$ has the left lifting property with respect to all trivial fibrations. Consider the following square where p is at every point a trivial fibration

$$\begin{array}{ccc} F_X^D & \longrightarrow & G \\ {\scriptstyle \widehat{i}}\downarrow & & \downarrow{\scriptstyle p} \\ F_{X'}^D & \longrightarrow & H \end{array}$$

Our goal is to construct a lift $F_{X'}^D \to G$, and the main point is that maps $F_{X'}^D \to G$ correspond with maps $X' \to G(D)$. For every object D' of $\mathbb{D}$ we can look at the square

$$\begin{array}{ccc} F_X^D(D') & \longrightarrow & G(D') \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_{D'}} \\ F_{X'}^D(D') & \longrightarrow & H(D') \end{array}$$

To find a lift, one might expect that we can take a lift at every point. However, together these might not be a natural transformation, and this is why we defined $F(D')$ as $\coprod_{\alpha \in \mathbb{D}(D,D')} X$. The idea is here to first find lift for copy of $F(D)$ at the identity arrow, and then to extend it.

Let us execute this plan for which we look at the square

$$\begin{array}{ccc} X & \longrightarrow & G(D) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_D} \\ X' & \longrightarrow & H(D) \end{array}$$

This diagram has a lift $\widetilde{h} : X' \to G(D)$. Now for an object D' and an arrow $f : D \to D'$ we define $X \to G(D')$ as $G(f) \circ \widetilde{h}$. Since $F_X^D(D') = \coprod_{\alpha \in \mathbb{D}(D,D')} X$, this is sufficient to define a map $h_{D'} : F_X^D(D') \to G(D')$ for all D'.

Next we need to check that $h_{D'}$ is natural. Suppose we have $f : D' \to D''$, and then we need to check the following diagram commutes

$$\begin{array}{ccc}
\coprod_{\alpha\in\mathbb{D}(D,D')} X & \xrightarrow{h_{D'}} & G(D') \\
\Big\downarrow{\scriptstyle F_X^D(f)} & & \Big\downarrow{\scriptstyle G(f)} \\
\coprod_{\alpha\in\mathbb{D}(D,D'')} X & \xrightarrow[h_{D''}]{} & G(D'')
\end{array}$$

To check it commutes, we check it for all factors of $\coprod_{\alpha\in\mathbb{D}(D,D')} X$, so let $g : D \to D'$ be arbitrary. Under $F_X^D(f)$ the gth factor of $\coprod_{\alpha\in\mathbb{D}(D,D')} X$ gets mapped to the $(f \circ g)$th factor of $\coprod_{\alpha\in\mathbb{D}(D,D'')} X$. So, it simplifies on this factor to the following rectangle

$$\begin{array}{ccccc}
X & \xrightarrow{\widetilde{h}} & G(D) & \xrightarrow{G(g)} & G(D') \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & & & & \Big\downarrow{\scriptstyle G(f)} \\
X & \xrightarrow[\widetilde{h}]{} & G(D) & \xrightarrow[G(f\circ g)]{} & G(D'')
\end{array}$$

which commutes by functoriality of G. Hence, h is indeed a natural transformation, and thus we have found the desired lift.

Similarly, if $j : X \to X'$ is a trivial cofibration, then the map $\widetilde{j} : F_X^D \to F_{X'}^D$ is a trivial cofibration. Motivated by this we define $I = \{\widetilde{i} \mid i \in I_{\mathcal{M}}\}$ and $J = \{\widetilde{j} \mid j \in J_{\mathcal{M}}\}$. Next we show (3). Let $p : G \to H$ be any map which has the right lifting property with respect to all maps in I. We need to show that $p_{D'} : G(D') \to H(D')$ is a trivial fibration meaning that is has the right lifting property with respect to all maps in I. So, take an arbitrary map $i : X \to X'$ with $i \in I$, and an arbitrary square

$$\begin{array}{ccc}
X & \longrightarrow & G(D') \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle p_{D'}} \\
X' & \longrightarrow & H(D')
\end{array}$$

We can extend this to a square on the presheaves by using similar techniques as before. The maps in the square determine the natural transformations on the identity factor, and then we can extend them. This thus gives the following commuting square

$$\begin{array}{ccc}
F_X^{D'} & \longrightarrow & G \\
\Big\downarrow{\scriptstyle \widetilde{i}} & & \Big\downarrow{\scriptstyle p} \\
F_{X'}^{D'} & \longrightarrow & H
\end{array}$$

By assumption we have a lift $h : F_{X'}^D \to H$. Evaluating at D' gives the rectangle

$$\begin{array}{ccccc}
X & \longrightarrow & \coprod_{\alpha\in\mathbb{D}(D',D')} X & \longrightarrow & G(D') \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \widetilde{i}} & & \Big\downarrow{\scriptstyle p_{D'}} \\
X' & \longrightarrow & \coprod_{\alpha\in\mathbb{D}(D',D')} X' & \longrightarrow & H(D')
\end{array}$$

where the inclusions are to the Id -factor. Hence, p is a trivial fibration, and thus (3) holds.

Checking (2) is easy now. Since $J \subseteq I$, we have $\mathrm{Cof}(J) \subseteq \mathrm{Cof}(I)$. Every map $j \in J$ has the left lifting property with respect to the fibrations, and with a similar argument we can show that this implies that j is a weak equivalence at every point. Also, since operations of Cof are performed pointwise, and every map in J is a weak equivalence at every point, we have $\mathrm{Cof}(J) \subseteq We$, because it holds in $\mathscr{M}$ and weak equivalences are defined pointwise.

Dually. one has the *injective model structure* where the cofibrations and weak equivalences are defined pointwise [**Hel88**]. However, to show that this is a model structure, requires more technique. Since we do not need this model structure in this thesis, we will not look at the proof.

On the other, there is a small number of examples in which we do not need the small object argument. For example, we could consider the *Reedy model structure* which again is on functor categories $\mathscr{M}^{\mathbb{D}}$. However, in this case we require $\mathbb{D}$ to be a special kind of category, namely a *Reedy category*.

Definition 3.1.6 (Reedy Category)**.** A *Reedy category* is a triple $(\mathbb{D}, \mathbb{D}_+\mathbb{D}_-)$ where $\mathbb{D}$ is a small category with two subcategories $\mathbb{D}_+$ and $\mathbb{D}_-$ with a function $d : \mathbb{D} \to \lambda$ sending objects of $\mathbb{D}$ to some ordinal $\mu < \lambda$. This data is required to satisfy the following

(1) Every map f can be factored uniquely as $g \circ h$ where $g \in \mathbb{D}_+$ and $h \in \mathbb{D}_-$.
(2) If we have a nonidentity map $f : A \to B$ in $\mathbb{D}_+$, then $d(A) < d(B)$.
(3) If we have a nonidentity map $f : A \to B$ in $\mathbb{D}_-$, then $d(A) > d(B)$.

The simplex category Δ is a Reedy category where the degree of $[n]$ is n. We have subcategories $\mathbb{D}_+$ consisting of the monomorphisms, and $\mathbb{D}_-$ which contains the epimorphisms. To construct the model structure on $\mathscr{M}^{\mathbb{D}}$ where $\mathbb{D}$, we will use transfinite induction. For this the main ingredients are the *matching spaces* and the *latching spaces*. To define the latching space, first we define a category $\mathbb{D}_{+,X}$ whose objects are arrows nonidentity arrows $f : Y \to X$, and the arrows from $f : Y \to X$ to $g : Z \to X$ are maps $h : Y \to Z$ such that $f = g \circ h$. The latching space functor L_X is defined to be the composition

$$\mathscr{C}^{\mathbb{D}} \longrightarrow \mathscr{C}^{\mathbb{D}_{+,X}} \xrightarrow{\mathrm{colim}} \mathscr{C}$$

where the first map is given by restriction. Dually, we can define the matching space. For an object X we first define a category $\mathbb{D}_{-,X}$ whose objects are nonidentity arrows $f : X \to Y$ and the arrows are again commutative triangles. The matching space functor M_X is then defined to be the composition

$$\mathscr{C}^{\mathbb{D}} \longrightarrow \mathscr{C}^{\mathbb{D}_{-,X}} \xrightarrow{\lim} \mathscr{C}$$

From the universal property of the limit and the colimit we always get natural transformations $L_X(A) \to A_X \to M_X(A)$ for X in $\mathbb{D}$ and $A \in \mathscr{C}^{\mathbb{D}}$.

Example 3.1.7. The cofibrations and fibrations are defined in a different way which gives this model structure different uses from the injective and projective model structure. A natural transformation $\eta : X \Rightarrow Y$ is called a *Reedy cofibration* iff every map $X_i \coprod_{L_i X} L_i Y \to Y_i$ is a cofibration, and f is called a *Reedy fibration* iff every $X_i \to$

$Y_i \times_{M_i Y} M_X$ is a fibration. Here $X_i \coprod_{L_i X} L_i Y$ is the pushout of the diagram

$$\begin{array}{ccc} L_i X & \longrightarrow & L_i Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_i \coprod_{L_i X} L_i Y \end{array}$$

and $Y_i \times_{M_i Y} M_X$ is the pullback

$$\begin{array}{ccc} Y_i \times_{M_i Y} M_X & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ M_i X & \longrightarrow & M_i Y \end{array}$$

This way we get a model structure on $\mathscr{M}^{\mathbb{D}}$ for a Reedy category $\mathbb{D}$. Instead of Quillen's small object argument, one can use transfinite induction to show the lifting axioms and the factorization axioms, and a precise proof can be found in [**Hov07**].

The disadvantage of Reedy categories is that they cannot have nontrivial automorphisms. If f is a nontrivial automorphism, then we cannot have $f \in \mathbb{D}_+$ or $f \in \mathbb{D}_-$, because f neither raises nor lowers the degree. If we had such a map f, then using the factorizations we can write $f = g \circ h$ and $h \circ f^{-1} = g' \circ h'$ with $g, g' \in \mathbb{D}_+$ and $h, h' \in \mathbb{D}_-$. Now we can compute

$$\mathrm{Id} = g \circ h \circ f^{-1} = g \circ g' \circ h'$$

where $g, g' \in \mathbb{D}_+$ and $h' \in \mathbb{D}_-$. Since the factorization is unique, we have $g \circ g' = \mathrm{Id}$ and $h' = \mathrm{Id}$. The map g lies in $\mathbb{D}_+$, so if it would not be the identity, then it raises the degree. However, if g raises the degree, then $g \circ g'$ must raise the degree as well, but this is impossible. The map $g \circ g'$ is the identity, so it does not raise the degree. This allows us to conclude that $g = \mathrm{Id}$, and thus $f = g \circ h = h$. By definition of h we have $h \in \mathbb{D}_-$, and that gives $f \in \mathbb{D}_-$. This is absurd, because f can neither raise nor lower the degree, and thus a Reedy category cannot have nontrivial automorphisms.

To solve this, one can consider the *generalized Reedy model structure* from [**BM11**] The definitions of the model structure and the main idea of the proof is similar, but it is more complicated. We will not require generalized Reedy model structures in this thesis, and thus we will not discuss them.

3.2. Transfering Model Structures

Transfer basically says that with adjunctions we can create new model structures under suitable assumptions. The conditions of the proposition are chosen in such a way that we can perform the small object argument in the new category by doing it in the old category. The following proposition is from [**Cra95**], but the ideas originally came from [**Qui67**].

Proposition 3.2.1 (Transfer). *Let $\mathscr{C}$ be a category with all small limits and colimits and let $\mathscr{M}$ be a model category. Suppose that we have an adjunction $L \dashv R$ with $L : \mathscr{M} \to \mathscr{C}$ and that the following conditions are satisfied*

(1) *$\mathscr{M}$ is combinatorial and the cofibrations and trivial cofibrations are generated by I and J respectively.*
(2) *$\mathscr{C}$ is locally presentable.*

(3) *Weak equivalences in $\mathscr{M}$ are closed under filtered colimits.*
(4) *The right adjoint R preserves filtered colimits.*
(5) *Given a map $f \in J$ and a pushout g of $L(f)$, the map $R(g)$ is a weak equivalence.*

Then there is a combinatorial model structure on $\mathscr{C}$ where the weak equivalences are $R^{-1}(\mathrm{We})$*, the fibrations are* $R^{-1}(\mathrm{Fib})$*, and the cofibrations are the maps with the left lifting property with respect to* $R^{-1}(\mathrm{We}) \cap R^{-1}(\mathrm{Fib})$.

PROOF. Since $\mathscr{C}$ satisfies (M1) by assumption. If two of g, f and $g \circ f$ are weak equivalences, then two of $R(g)$, $R(f)$ and $R(g \circ f)$ are weak equivalences. The third must then also be a weak equivalence, and thus (M2) is satisfied. For (M3) we can do the same thing: if f is a retract of g, then $R(f)$ is a retract of $R(g)$, and because the cofibrations are defined as maps having the left lifting property with respect to some class, they are closed under retracts too. By definition the cofibrations have the left lifting property with respect to the trivial fibrations, and this gives one half of (M4).

Now we continue with (M5), and we apply the small object argument on $L(I)$ (recall that the cofibrations are generated by I). Before we do that, we need to play with the adjunction a bit. First note that for $i \in I$ the map $L(i)$ is a cofibration. For that we start with the diagram

$$\begin{array}{ccc} L(A) & \longrightarrow & B \\ {\scriptstyle L(i)}\downarrow & & \downarrow{\scriptstyle g} \\ L(C) & \longrightarrow & D \end{array}$$

where $g \in R^{-1}(We) \cap R^{-1}(Fib)$. Next we factor this diagram using the counit map

$$\begin{array}{ccccc} L(A) & \longrightarrow & L(R(B)) & \longrightarrow & B \\ {\scriptstyle L(i)}\downarrow & & \downarrow{\scriptstyle L(R(g))} & & \downarrow{\scriptstyle g} \\ L(C) & \longrightarrow & L(R(D)) & \longrightarrow & D \end{array}$$

and to solve the lifting problem, we look for a lift from $L(C) \to L(R(B))$. To find this lift, we first find a good map from $C \to R(B)$, and then we apply L on it. For that we look at the diagram

$$\begin{array}{ccc} A & \longrightarrow & R(B) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle R(g)} \\ C & \longrightarrow & R(D) \end{array}$$

Note that $R(g) \in We \cap Fib$ and that i is a cofibration. Hence, the desired lift exist, and this gives the lift we wanted to show that $L(i)$ has the left lifting property with respect to $R^{-1}(We) \cap R^{-1}(Fib)$.

Let f be a map which has the right lifting property with respect to all $L(i)$. We shall prove that $R(f) \in We \cap Fib$, and for that we need to solve the following lifting problem

$$\begin{array}{ccc} A & \longrightarrow & R(B) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle R(f)} \\ C & \longrightarrow & R(D) \end{array}$$

Again we factor this diagram, but this time we use the unit map

$$\begin{CD} A @>>> R(L(A)) @>>> R(B) \\ @V{i}VV @VV{R(L(i))}V @VV{R(f)}V \\ C @>>> R(L(C)) @>>> R(D) \end{CD}$$

To solve this lifting problem, we find a lift from $R(L(C))$ to $R(B)$, and we do this by finding a suitable map $L(C) \to B$. Look at the following diagram

$$\begin{CD} L(A) @>>> B \\ @V{L(i)}VV @VV{f}V \\ L(C) @>>> D \end{CD}$$

Since f has the right lifting property with respect to all $L(i)$, this diagram has a lift and thus $R(f)$ has the right lifting property with respect to I.

The small object argument gives that every map f can be factored as a $L(I)$-cellular complex $\alpha(f)$ followed by a map $\beta(f)$ having the right lifting property with respect to $L(I)$. Since all $L(i)$ for $i \in I$ are cofibrations and the cofibrations are closed under the operations to form cellular complexes, we get that $\alpha(f)$ has the left lifting property with respect to $R^{-1}(We) \cap R^{-1}(Fib)$. Also, $\beta(f)$ has the right lifting property with respect to $L(I)$, and from that we conclude that $R(\beta(f)) \in We \cap Fib$. This allows us to conclude that $\beta(f)$ is a trivial fibration, and thus we have found one of the desired factorizations.

To find the other factorizations, we apply the same thing. We apply the small object argument to $L(J)$ to see that we can factorize f as a map $\gamma(f)$ which is a $L(J)$-cellular complex followed by a map $\delta(f)$ which has the right lifting property with respect to $L(J)$. By using the same argument as for the trivial fibrations we can show that $\delta(f)$ is a fibration. For $\gamma(f)$ there is an issue: the argument we used before can only be used to show that it is a cofibration, but it should be a weak equivalence as well. To show that it is a weak equivalence as well, we use the additional assumptions. A cellular complex on J is a transfinite composition of pushouts of maps in J, and we need to prove that its image under R is a weak equivalence. Since R preserves filtered colimits and thus transfinite compositions, it is sufficient to prove that the pushouts are mapped to weak equivalences. So, we have a pushout g of $L(i)$, and we need to prove that $R(g)$ is a weak equivalence. This is precisely assumption (5), and thus (M5) holds.

To finish the proof, we need to show the other half of (M4) states that fibrations have the right lifting property with respect to trivial cofibrations. Let i be a trivial cofibration, and we just showed that we can factor i as a $L(J)$-complex j followed by a fibration p. This means that we have the following diagram

$$\begin{CD} A @>{j}>> B \\ @V{i}VV @VV{p}V \\ C @>>{\mathrm{Id}}> C \end{CD}$$

Since i and j are weak equivalences, p must be a weak equivalence as well. Hence, i is a trivial cofibration and p is a trivial fibration, and thus we get a lift $h : C \to B$. This gives that i is a retract of j, and thus i is a $L(J)$ cofibration, because it is the retract of a $L(j)$-complex. Since $L(J)$-cofibrations have the left lifting property with respect to fibrations, the other half of (M4) follows. □

Transfer can be used to find a model a structure on simplicial universal algebras **SAlg**. We have a forgetful functor i : **SAlg** → **SSet**, and this has a left adjoint F : **SSet** → **SAlg** which takes the free algebra. Since i is the forgetful functor, it preserves all colimits. Later we show that (3) and (5) hold as well, and thus we get a model structure on the simplicial algebras in which the fibrations and weak equivalences are detected by the forgetful functor.

3.3. Homotopy Colimits

In normal category theory we can glue objects by using colimits. However, this construction is not nice from a homotopical perspective, because it is not invariant under homotopy. We could for example glue two lines to obtain a circle via the following colimiting diagram

$$\begin{array}{ccc} S^0 & \longrightarrow & I \\ \downarrow & & \downarrow \\ I & \longrightarrow & S^1 \end{array}$$

But the spaces $*$ and I are homotopy equivalent, and via this homotopy equivalence we can get the following colimit

$$\begin{array}{ccc} S^0 & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

Since S^1 and $*$ are not homotopy equivalent, colimits are not invariant under homotopy.

This is quite a disadvantage for algebraic topologists, because it does not allow for a computation of the homotopy groups of the colimit. To fix this, we replace colimits by homotopy colimits which will turn out to be invariant under homotopy. Instead of gluing, we glue up to homotopy, and to formalize this, we need the language of *cosimplicial resolutions*. Recall that a cosimplicial object of $\mathscr{C}$ is an element of $\mathscr{C}^{\Delta}$, and because Δ is a Reedy category, we can put the Reedy model structure on this category by Example 3.1.7. The model category of cosimplicial resolutions in $\mathscr{M}$ with the Reedy model structure will be written as $c\mathscr{M}$. Also, for every object X of $\mathscr{C}$ we get a cosimplicial object $c^*(X)$ which is X in every degree.

Definition 3.3.1 (Cosimplicial Resolutions)**.** Let X be an object of a model category $\mathscr{M}$. Then a *cosimplicial resolution* is a Reedy cofibrant object Γ in $\mathscr{M}^{\Delta}$ with a degreewise weak equivalence $\Gamma \to c^*(X)$.

Let us try to visualize what cosimplicial resolutions are. Because we have the weak equivalence $\Gamma \to c^*(X)$, every Γ^i is equivalent to X. The resolution Γ looks as follows

$$\Gamma^0 \rightrightarrows \Gamma^1 \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \cdots$$

But we know more: we know that $! : * \to \Gamma$ is a cofibration. To see what this means, we need to work out the definitions of Example 3.1.7. By definition ! is a cofibration iff for all degree n the map $L_n\Gamma \coprod_{L_n *} * \to \Gamma_n$ is a cofibration. Since $*$ is the initial object, this means that the map $L_n\Gamma \to \Gamma$ must be a cofibration. For $L_n\Gamma$ we glue Γ_{n-1} according to the boundary of Δ^n, and the map $L_n\Gamma \to \Gamma$ is the boundary inclusion. Hence, with a cosimplicial resolution we can capture the homotopical data of an object. Next we extend our definition of cosimplicial resolution to talk about cosimplicial resolutions of functors.

Definition 3.3.2. Given a functor $\gamma : \mathscr{C} \to \mathscr{M}$, then a cosimplicial resolution of γ is a functor $\Gamma : \mathscr{C} \to c\mathscr{M}$ with a weak equivalence $\Gamma(X) \to c^*(\gamma(X))$ such that each $\Gamma(X)$ is Reedy cofibrant.

Using the injective model structure on $(c\mathscr{N})^{\mathscr{C}}$, one can state the definition more compactly by saying that Γ is cofibrant. At every point we thus have a cosimplicial resolution of $\gamma(X)$, and the weak equivalence must be natural. Let Γ_1 and Γ_2 be cosimplicial resolutions of γ. Then a map from Γ_1 to Γ_2 is a natural transformation $\eta : \Gamma_1 \Rightarrow \Gamma_2$ such that the following diagram commutes

$$\begin{array}{ccc} \Gamma_1(X) & \xrightarrow{\eta_X} & \Gamma_2(X) \\ & \searrow \quad \swarrow & \\ & c^*(\gamma(X)) & \end{array}$$

So, this gives that for every γ we have a category of cosimplicial resolutions of γ, and we denote this category by $\mathrm{coRes}(\gamma)$. An easy property of cosimplicial resolutions is that they always exist.

Proposition 3.3.3. *Let $\mathscr{M}$ be a model category, and let $\gamma : \mathscr{C} \to \mathscr{D}$ be a diagram. Then γ has a cosimplicial resolution Γ.*

We will not give the precise proof, but instead a sketch and leave the details to the reader. The main idea of the proof is to take a cofibrant replacement in the Reedy model structure. This way we can find cosimplicial resolutions of any given object. Because the factorizations are functorial, we can turn this into a functor. Hence, this results in a cosimplicial resolution of γ.

The next step in defining colimits, is realizing simplicial sets using a cosimplicial resolution γ. For a set S and an object X we write $S \cdot X$ for $\coprod_{s \in S} X$. Using this notation we define the left action $K \otimes_\Delta \gamma$ for a cosimplicial resolution γ and a simplicial set K as the following coequalizer.

$$K \otimes_\Delta \gamma = \mathrm{coeq}\left(\coprod_{[k] \to [m]} K_m \cdot \gamma^k \rightrightarrows \coprod_n K_n \cdot \gamma^n \right)$$

The main idea of this definition is that we realize γ^n as the n-simplex, and then glue according to the simplicial set. So, first we put all n-simplices together using the big coproduct $\coprod_n K_n \cdot \gamma^n$. Now we only need to glue them in the correct way by identifying the boundaries and the degeneracies in the correct way. That is why we have the coequalizer and we sum over all arrows $[k] \to [m]$ in the first coproduct. Also, note the similarities between this formula and the formula for the geometric realization of a simplicial set. They are the same formulas, but with n-simplex replaced by γ^n. Let us compute $\Delta[n] \otimes_\Delta \gamma$.

Proposition 3.3.4. *For a cosimplicial resolution γ we have $\Delta[n] \otimes_\Delta \gamma = \gamma^n$.*

PROOF. By definition we have

$$\Delta[n] \otimes_\Delta \gamma = \operatorname{coeq}\left(\coprod_{[k]\to[m]} \Delta[n]_m \cdot \gamma^k \rightrightarrows \coprod_k \Delta[n]_k \cdot \gamma^k \right),$$

and we have to show this coequalizer is γ^n. For that we use the universal property of the coequalizer. Note that the object $\Delta[n]_k \cdot \gamma^k$ has for every map $f : [k] \to [n]$ a copy of γ^k, and to define a map $\Delta[n]_k \cdot \gamma^k \to \gamma^n$ we need to make a map $\gamma^k \to \gamma^n$ for every $f : [k] \to [n]$. Since γ is a cosimplicial resolution, we always have such a map, namely $\gamma(f) : \gamma^k \to \gamma^n$. This allows us to define $\coprod_k \Delta[n]_k \cdot \gamma^k \to \gamma^n$, because on the pair (n, f) we define it as $\gamma(f)$.

Next we show that this map makes the required diagram commute. We need to check this on every copy $\Delta[n]_m \cdot \gamma^k$ for $g : [k] \to [m]$, and since $\Delta[n]_m \cdot \gamma^k$ is defined as $\coprod_{f:[m]\to[n]} \gamma^k$, it suffices to check it on every copy γ^k with $f : [m] \to [n]$ and $g : [k] \to [m]$. If we work out the definitions, then we get the following diagram

$$\begin{array}{ccc} \gamma^k & \xrightarrow{\mathrm{Id}} & \gamma^k \\ {\scriptstyle \gamma(g)}\downarrow & & \downarrow{\scriptstyle \gamma(f\circ g)} \\ \gamma^m & \xrightarrow[\gamma(f)]{} & \gamma^n \end{array}$$

and this commutes, because γ is functorial.

Lastly, we need to check γ^n satisfies the universal property. Suppose, we have a map $f : \coprod_k K_k \cdot \gamma^k \to Z$ such that $\coprod_{[k]\to[m]} \Delta[n]_m \cdot \gamma^k \rightrightarrows \coprod_k \Delta[n]_k \otimes_\Delta \gamma^k \to Z$ commutes. Note for the identity map $\mathrm{Id} : [n] \to [n]$ this gives the diagram

$$\begin{array}{ccc} \gamma^n & \xrightarrow{\mathrm{Id}} & \gamma^n \\ & {\scriptstyle g}\searrow & \\ & & Z \end{array}$$

and from this we can already conclude that the map $\gamma^n \to Z$ would be unique. Also, this already gives a candidate which we call g. To check that g has the right property, we need to check that for $h : [k] \to [n]$ the diagram

$$\begin{array}{ccc} \gamma^k & \xrightarrow{\gamma(h)} & \gamma^n \\ & {\scriptstyle f_h}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

This commutes, because f is a natural transformation. Hence, γ^n is indeed isomorphic to $\Delta[n] \otimes_\Delta \gamma$. □

Now we can define homotopy colimits. Recall from Example 2.1.4 that we have a functor $\mathscr{N}$, called the nerve functor, sending a small category $\mathscr{C}$ to the simplicial set $\mathscr{N}(\mathscr{C})$ where $\mathscr{N}(\mathscr{C})_n$ consists of all strings of n composable arrows. Also, recall that the under category $C \downarrow \mathscr{C}$ for a category $\mathscr{C}$ and object C is the category where the objects are arrows $C \to D$ and arrows from $f : D \to C$ to $g : E \to C$ are arrows $h : D \to E$ such that $f = g \circ h$.

Definition 3.3.5 (Homotopy Colimit)**.** Let $X : I \to \mathscr{M}$ be a diagram. Also, let Γ be a cosimplicial resolution of X. Then we define the *homotopy colimit* of X to be

$$\operatorname{hocolim}_I(X) = \operatorname{coeq}\left(\coprod_{i\to j} \mathscr{N}(j \downarrow I)^{\mathrm{op}} \otimes_\Delta \Gamma(i) \rightrightarrows \coprod_i \mathscr{N}(i \downarrow I)^{\mathrm{op}} \otimes_\Delta \Gamma(i)\right)$$

The choice of the cosimplicial resolution matters for $\operatorname{hocolim}_I(X)$, but in this thesis the chosen cosimplicial resolution will always be clear. When proving properties about homotopy colimits, we will assume the cosimplicial resolution to be fixed. One can show that the homotopy colimit is the left derived functor of the colimit, and a proof is given in Theorem 9.1 of [**Shu06**].

To motivate the definition, we will consider homotopy pushouts of topological spaces as an example. Recall that a pushout is the colimit of a diagram of the form

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ {\scriptstyle g}\downarrow & & \\ X_2 & & \end{array}$$

We can find a cosimplicial resolution for this diagram by taking $\Gamma(i)$ to be $X_i^{\mathrm{Cof}} \times \Delta^n$ in degree n, and remember that the cofibrant replacement of topological spaces is given by the mapping cylinder. Taking the projection in every degree gives a degreewise weak equivalence to $c^*(X)$. Also, we need the map $L_n(\Gamma) \to \Gamma_n$ to be a cofibration, and this is so, because the boundary inclusion of $X \times \partial\Delta^n$ into $X \times \Delta^n$ is a cofibration for every space X. Note that the under categories $1 \downarrow I$ and $2 \downarrow I$ are trivial, and that $0 \downarrow I$ is I itself.

Let us compute $\mathscr{N}((1 \downarrow I)^{\mathrm{op}}) \otimes_\Delta \Gamma(1)$ and $\mathscr{N}((2 \downarrow I)^{\mathrm{op}}) \otimes_\Delta \Gamma(2)$ first. By definition we have

$$\begin{aligned} \mathscr{N}((1 \downarrow I)^{\mathrm{op}}) \otimes_\Delta \Gamma(1) &= \operatorname{coeq}\left(\coprod_{[k]\to[m]} \Gamma(1)_k \rightrightarrows \coprod_n \Gamma(1)_n\right) \\ &= \operatorname{coeq}\left(\coprod_{[k]\to[m]} X_1^{\mathrm{Cof}} \times \Delta^k \rightrightarrows \coprod_n X_1^{\mathrm{Cof}} \times \Delta^n\right) \end{aligned}$$

and this is the geometric realization of the constant simplicial space which is X_1^{Cof} in every degree. Using Lemma 11.8 of [**May72**] this is homeomorphic to X_1^{Cof}. Similarly, we can show that $\mathscr{N}((2 \downarrow I)^{\mathrm{op}}) \otimes_\Delta \Gamma(2) = X_2$.

Also, we need to determine $\mathscr{N}((0 \downarrow I)^{\mathrm{op}}) \otimes_\Delta \Gamma(0)$. For that we first look at $X_0 \times K_m$ which by definition is

$$\begin{aligned} \coprod_{x\in K_m} X_0 \times \Delta^k &= (X_0 \times \Delta^k) \times (\coprod_{x\in K_m} *) \\ &= (X_0 \times \Delta^k) \times |\coprod_{f:[1]\to[m]} \Delta[0]| \\ &= (X_0 \times \Delta^k) \times |K_m| \\ &= (X_0 \times |K_m|) \times \Delta^k \end{aligned}$$

Here we regard the simplicial set as a discrete simplicial space. We can thus conclude that

$$\mathcal{N}((0 \downarrow I)^{\text{op}}) \otimes_{\Delta} \Gamma(1) = \text{coeq}\left(\coprod_{[k]\to[m]} (X_0^{\text{Cof}} \times |K_m|) \times \Delta^k \rightrightarrows \coprod_n (X_0^{\text{Cof}} \times |K_n|) \times \Delta^n \right)$$

and this is $X_0^{\text{Cof}} \times |K|$, because geometric realization commutes with finite limits by [**May99**]. Since $|\mathcal{N}((0 \downarrow I)^{\text{op}}|$ is homeomorphic to the interval Δ^1, we can conclude that $\mathcal{N}((0 \downarrow I)^{\text{op}}) \otimes_{\Delta} \Gamma(0) = X_0 \times I$. Also, the maps f and g give maps $X_0^{\text{Cof}} \to X_0^{\text{Cof}} \times \Delta^1$, and these maps are determined by the given homeomorphism. We can assume that for $f : X_0 \to X_1$ the induced map $X_0 \to X_0 \times \Delta^1$ is $\iota_0(x) = (x, 0)$ and for $g : X_0 \to X_2$ it is $\iota_1(x) = (x, 1)$.

The last step is gluing these pieces together which gives the homotopy colimit

$$\text{coeq}\left(\coprod_{i\to j} \mathcal{N}(j \downarrow I)^{\text{op}} \otimes_{\Delta} \Gamma(i) \rightrightarrows \coprod_i \mathcal{N}(i \downarrow I)^{\text{op}} \otimes_{\Delta} \Gamma(i) \right).$$

In the same way as before we can show that $\mathcal{N}(1 \downarrow I)^{\text{op}} \otimes_{\Delta} \Gamma(0) = \mathcal{N}(2 \downarrow I)^{\text{op}} \otimes_{\Delta} \Gamma(0) = X_0^{\text{Cof}}$. For this coequalizer we need to glue several things, and for that we need to look at the two nonidentity maps. First of all, we have $f : X_0^{\text{Cof}} \to X_1^{\text{Cof}}$ and $\iota_0 : X_0^{\text{Cof}} \to X_0^{\text{Cof}} \times \Delta^1$ mapping x to $(x, 0)$. Second of all, we have $g : X_0^{\text{Cof}} \to X_2^{\text{Cof}}$ and $\iota_1 : X_0^{\text{Cof}} \to X_0^{\text{Cof}} \times \Delta^1$ which maps x to $(x, 1)$. Hence, the homotopy pushout of this diagram is $(X_0^{\text{Cof}} \times \Delta^1) \coprod X_1^{\text{Cof}} \coprod X_2^{\text{Cof}}$ where we identify $(x, 0)$ with $f(x)$ and $(x, 1)$ with $g(x)$.

This explains what a homotopy colimit is. Instead of gluing the spaces, we glue up to homotopy. Sequences of arrows give the higher homotopies which can be understood in a similar fashion as in the previous example.

Next we discuss some properties of homotopy colimits. Suppose we have two diagrams $X_1, X_2 : I \to \mathcal{M}$ and a natural transformation $\eta : X_1 \Rightarrow X_2$. For X_1 and X_2 we can find cosimplicial resolutions Γ_1 and Γ_2, and because we have a map $\eta : X_1 \Rightarrow X_2$, we get a map $\Gamma_1 \to \Gamma_2$. Also, we get a map $\text{hocolim}_I X_1 \to \text{hocolim} X_2$, because we can define maps $\mathcal{N}(j \downarrow I)^{\text{op}} \otimes_{\Delta} \Gamma_2(i) \to \mathcal{N}(j \downarrow I)^{\text{op}} \otimes_{\Delta} \Gamma_2(i)$ and all these maps together give a map $\text{hocolim}_I X_1 \to \text{hocolim} X_2$. Now suppose that for every object i of I the map η_i is a weak equivalence. If that is the case, the map $\text{hocolim}_I X_1 \to \text{hocolim} X_2$ is a weak equivalence. The proof of this long, and is given in Theorem 18.5.1 from [**Hir00**].

The next property is about pulling back diagrams. Suppose we have two categories I_1, I_2, a functor $f : I_1 \to I_2$ and a functor $X : I_2 \to \mathcal{M}$. Now we can define $f^*(X) : I_1 \to \mathcal{M}$ as $X_1(i) = X_2(f(i))$. For X we can find a cosimplicial resolution Γ, and Γ can be pulled back to a cosimplicial resolution for $f^*(X)$. Namely, we can define $(f^*(\Gamma)(i))_n = \Gamma(f(i))_n$. Since $f^*(\Gamma(i))_n = \Gamma(f(i))_n$, we get an arrow $f^*(\mathcal{N}(j \downarrow I_1)^{\text{op}} \otimes_{\Delta} \Gamma(i)) \to \mathcal{N}(j \downarrow I_2)^{\text{op}} \otimes_{\Delta} \Gamma(f(i))$, and all these arrows together give an arrow $f_* : \text{hocolim}_{I_1} f^*(X) \to \text{hocolim}_{I_2} X$. This is functorial meaning that for functors $f : I_1 \to I_2$ and $g : I_2 \to I_3$ we ave $g_* \circ f_* = (g \circ f)_*$ and this follows directly from the formulas.

For another property we need natural transformations. Suppose, we have $f, g : I_1 \to I_2$, a diagram $X : I_2 \to \mathcal{M}$, and a natural transformation $\eta : f \Rightarrow g$. Object i of I_1 then give an arrow $\eta_i : f(i) \to g(i)$, and that way we can make an arrow $\text{hocolim}_{I_1} f^*(X) \to \text{hocolim}_{I_1} g^*(X)$. With additional techniques which we did not discuss in this thesis, one can prove that the following diagram commutes in the homotopy category

$$\begin{array}{ccc} \operatorname{hocolim}_{I_1}(f^*)(X) & \xrightarrow{f_*} & \operatorname{hocolim}_{I_2} X \\ \downarrow{\scriptstyle \eta_*} & \nearrow{\scriptstyle g_*} & \\ \operatorname{hocolim}_{I_1}(g^*)(X) & & \end{array}$$

Using all these properties we can deduce the following proposition from [**Dug01a**] which we can use to recognize whether two homotopy colimits over different indexing diagrams are weakly equivalent.

Proposition 3.3.6. *Given are two small categories I and J and a diagram $X : I \to \mathscr{M}$. Suppose that we have functors $f : I \to J$ and $g : J \to I$ with natural transformations $\theta : f \circ g \Rightarrow \mathrm{Id}$ and $\eta : g \circ f \Rightarrow \mathrm{Id}$ such that the following requirements are satisfied*

(1) *The maps $X(\eta_i) : X(g(f(i))) \to X(i)$ are weak equivalences.*
(2) *The maps $X(g(\theta_j)) : X(g(f(g(j)))) \to X(g(j))$ are weak equivalences.*

Then the map g_* is a weak equivalence.

PROOF. To show that g_* is a weak equivalence, we use a categorical fact. Suppose that we have maps $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$. such that $b \circ a$ and $c \circ b$ are isomorphisms. This means that $b \circ a$ has an inverse f and $c \circ b$ has an inverse g. Then we have $g \circ c \circ b = \mathrm{Id}$ and $b \circ a \circ f = \mathrm{Id}$, so b has an inverse. As a consequence of this we get $f \circ b \circ a = \mathrm{Id}$ and

$$a \circ f \circ b = b^{-1} \circ (b \circ a \circ f) \circ b = b^{-1} \circ \mathrm{Id} \circ b,$$

so b has an inverse. In the same way we can show that c has an inverse.

We have the following diagram

$$\operatorname{hocolim}_J (g \circ f \circ g)^* X \xrightarrow{g_*} \operatorname{hocolim}_I (g \circ f)^* X \xrightarrow{f_*} \operatorname{hocolim}_J g^* X \xrightarrow{g_*} \operatorname{hocolim}_I X$$

The maps are all induced by the functors f and g. To show that g_* and f_* are weak equivalences, it is sufficient to show that these are isomorphisms in the homotopy category. Hence, it suffices to show that the compositions are weak equivalences.

Let us start with the first one. Recall that we have a weak equivalence $X(\eta_i) : X(g(f(i))) \to X(i)$, and this gives the following diagram which commutes in the homotopy category

$$\begin{array}{ccccc} \operatorname{hocolim}_I (g \circ f)^* X & \xrightarrow{f_*} & \operatorname{hocolim}_J g^* X & \xrightarrow{g_*} & \operatorname{hocolim}_I X \\ \downarrow{\scriptstyle \eta_*} & & & \nearrow{\scriptstyle \mathrm{Id}_*} & \\ \operatorname{hocolim}_I \mathrm{Id}^*(X) & & & & \end{array}$$

By assumption the map η is an objectwise weak equivalence, so η is a weak equivalence. The map Id_* is a weak equivalence as well, and thus $g_* \circ f_*$ is a weak equivalence by the 2-out-of-3 property.

Now we do the second diagram, and here we use that the maps $X(g(\theta_j))$ are weak equivalences.

$$\begin{array}{ccccc} \operatorname{hocolim}_J (g \circ f \circ g)^* X & \xrightarrow{g_*} & \operatorname{hocolim}_I (g \circ f)^* X & \xrightarrow{f_*} & \operatorname{hocolim}_J g^* X \\ \downarrow{\scriptstyle (g\theta)_*} & & & \nearrow{\scriptstyle \mathrm{Id}_*} & \\ \operatorname{hocolim}_J g^*(X) & & & & \end{array}$$

Again the maps Id_* and $(g\theta)_*$ are weak equivalences. That $(g\theta)_*$ is a weak equivalence, is because it is an objectwise weak equivalence by assumption and thus a weak equivalence if you take homotopy colimits. □

This proposition also holds if we do not have a natural transformation $f \circ g \Rightarrow \mathrm{Id}$ or $g \circ f \Rightarrow \mathrm{Id}$, but rather zig-zags of natural transformations which get mapped to weak equivalences.

Let us discuss a quick application of this proposition for which we need the notion of a *contractible* category. A category $\mathscr{C}$ is called contractible if the geometric realization of its nerve is a contractible topological spaces. If $\mathscr{C}$ is a category, then we can build a simplicial set $\mathscr{N}(\mathscr{C})$ which in degree n is the set of all strings $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots \xrightarrow{f_n} A_{n+1}$ of n composable arrows. Also, if K is a simplicial set, then we can construct a topological space $|K|$ with the following formula

$$|K| = \operatorname{coeq}\left(\coprod_{i \to j} K(i) \times \Delta^j \rightrightarrows \coprod_i K(i) \times \Delta^i \right)$$

where Δ^i is the i-simplex and $K(i)$ is seen as a discrete topological space. So, shortly said, a category is contractible if $|\mathscr{N}(\mathscr{C})|$ is contractible.

Definition 3.3.7 (Homotopy Cofinal)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Define the category $X \downarrow F$ whose objects are arrows $X \to F(Y)$ and whose arrows are commutative triangles in the obvious way. Then we say that F is *homotopy cofinal* iff $\mathscr{N}(X \downarrow F)$ is a contractible category for all X

The important property of homotopy cofinal functors is given by the following corollary.

Corollary 3.3.8. *If $F : \mathscr{C} \to \mathscr{D}$ is homotopy cofinal and $X : \mathscr{D} \to \mathscr{M}$ is a diagram, then* $\operatorname{hocolim}_{\mathscr{C}} F^*(X) \to \operatorname{hocolim}_{\mathscr{D}} X$ *is a weak equivalence.*

To prove this, we need Quillen's Theorem A [**Qui73**] which says that $|F|$ is a homotopy equivalence if F is homotopy cofinal. Because both the nerve functor and the geometric realization functor are faithful, the conditions from Proposition 3.3.6 follow, and thus we can conclude that $\operatorname{hocolim}_{\mathscr{C}} F^*(X) \to \operatorname{hocolim}_{\mathscr{D}} X$ is a weak equivalence.

3.4. Bousfield Localization

The goal of *Bousfield localization* is to add weak equivalences to some model structure. This does not have an obvious solution, because we cannot just redefine the weak equivalences and then use the same cofibrations and fibrations. This is because the lifting property (M4) from Definition 2.1.1 might be violated. If there are more weak equivalences, then there might be more trivial fibrations, and this leads to problems.

Before giving concrete definitions, let us try to motivate this construction by recalling localization from commutative algebra. If we want to study an affine variety locally, then the tool we use is localization. From the coordinate ring we pass to a local ring, and this is done by formally adding inverses for some elements. Such constructions can also be done in other things like model categories. Instead of adding multiplicative inverses, we do something weaker. One would expect that some maps will be turned into isomorphisms, but instead we do something more homotopical. A collection of maps are made into weak equivalences. This way we can define the notion of a *S-localization*

Definition 3.4.1 (S-localization)**.** Let $\mathscr{M}$ be a model category and let S be a collection of maps in $\mathscr{M}$. An *S-localization* of $\mathscr{M}$ is a model category $\mathscr{M}/S$ with a left Quillen functor $F : \mathscr{M} \to \mathscr{M}/S$ such that $\mathbb{L}F$ maps arrows in S to weak equivalences. Also, $\mathscr{M}/S$ should satisfy a universal property namely that for all model categories $\mathscr{N}$ and left Quillen functors $G : \mathscr{M} \to \mathscr{N}$ such that $\mathbb{L}G$ maps S to the weak equivalences, we have a unique left Quillen functor $H : \mathscr{M}/S \to \mathscr{N}$ which makes the following diagram commute

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{G} & \mathscr{N} \\ {\scriptstyle F}\big\downarrow & \nearrow_{H} & \\ \mathscr{M}/S & & \end{array}$$

Unlike the situation in commutative algebra here localizations need not to exist. Also, it might be difficult to deal with them, because we do not know how to construct such localizations. If suitable assumptions are satisfied, then it is always possible to produce such a localization in such a way that we know the category, the weak equivalences, and the cofibrations. To define Bousfield localizations, we need some build up: we need the notion of a *homotopy function complex*

Normally, the sets $\mathrm{Mor}(X, Y)$ do not carry extra structure, but we would like them to be simplicial sets. In degree 0 we have the functions, in degree 1 we have the homotopies between functions, and so on. One way to get such a structure, is by replacing X by a cosimplicial object or Y by a simplicial object. Therefore, there are multiple paths to make it a simplicial set.

The first way gives the notion of a *left homotopy function complex*. We take a cosimplicial resolution for X and we replace Y by a fibrant object, and then we can construct the mapping space.

Definition 3.4.2 (Left Homotopy Function Complex)**.** Let $\mathscr{M}$ be a model category and let X and Y be objects. For any cosimplicial resolution Γ^* of X and fibrant approximation $\widehat{Y}$ of Y we say that the simplicial set

$$\mathrm{Map}(X, Y)_n = \mathscr{M}(\Gamma^n, \widehat{Y})$$

is a *left homotopy function complex* from X to Y.

Dually, we can define *right homotopy function complexes*, but for that we need a dual notion of cosimplcial resolution. For a cosimplicial resolution we look at $\mathscr{M}^{\Delta}$, so for a simplicial resolution we look at $\mathscr{M}^{\Delta^{op}}$.

Definition 3.4.3 (Simplicial Resolutions)**.** Let X be an object of a model category $\mathscr{M}$. Then a *simplicial resolution* is a Reedy fibrant object Γ in $\mathscr{M}^{\Delta^{op}}$ with a degreewise weak equivalence $\Gamma \to c_*(X)$ where $c_*(X)$ is the constant simplicial object.

Now we have enough to define right homotopy function complexes.

Definition 3.4.4 (Right Homotopy Function Complex)**.** Suppose that we have a model category $\mathscr{M}$ and objects X and Y. For any cofibrant approximation $\widetilde{X}$ of X and simplicial resolution Γ of Y we say that the simplicial set

$$\mathrm{Map}(X,Y)_n = \mathscr{M}(\widetilde{X},\Gamma_n)$$

is a *right homotopy function complex* from X to Y.

But there is also a third way, because we can also take a cosimplicial resolution for X and a simplicial resolution for Y. That way we get a bisimplicial set, and its diagonal will be the *two-sided homotopy function complex*.

Definition 3.4.5 (Two-Sided Homotopy Function Complex)**.** Given are a model category $\mathscr{M}$ and objects X and Y. Also, suppose we have any cosimplicial resolution Γ of X and a simplicial resolution Γ' of Y. Then the diagonal of the bisimplicial set defined as

$$\mathrm{Map}(X,Y)_{n,m} = \mathscr{M}(\Gamma^n,\Gamma'_m)$$

is said to be a *two-sided homotopy function complex* from X to Y.

Now we define a *homotopy function complex* from X to Y to be either a left homotopy function complex, right homotopy function complex or a two-sided homotopy function complex from X to Y. One important property of homotopy function complexes is that we can detect weak equivalences with them.

Proposition 3.4.6. *Let $\mathscr{M}$ be a model category and let $g : A \to B$ be a map. Then g is a weak equivalence iff for every fibrant object X the map* $\mathrm{Map}(A,X) \to \mathrm{Map}(B,X)$ *is a weak equivalence of simplicial sets.*

For the proof of this proposition we refer the reader to Theorem 17.7.7 in [**Hir00**]. This proposition will be the basis of defining Bousfield localizations. Some objects might detect all maps in the set S to be weak equivalences, and these objects will are called S-local. So, a fibrant object X is called *S-local* iff for all maps $g : A \to B$ in S the map $\mathrm{Map}(A,X) \to \mathrm{Map}(B,X)$ is a weak equivalence of simplicial sets. The point is that the S-local objects determine the weak equivalences in the localized model category. A *S-local equivalence* is a map $g : A \to B$ such that for all S-local objects the map $\mathrm{Map}(A,X) \to \mathrm{Map}(B,X)$ is a weak equivalence.

Definition 3.4.7 (Left Bousfield S-localization)**.** Let $\mathscr{M}$ be a model category, and let S be a set of maps in $\mathscr{M}$. Then a *left Bousfield S-localization* of M is a model category $\mathscr{M}/S$ with the same objects and cofibrations as $\mathscr{M}$ and whose weak equivalences are the S-local equivalences.

Left Bousfield S-localizations are indeed S-localizations, but they do not have to exist. However, in many examples they do exist. and their is a check for their existence.

Definition 3.4.8 (Left Proper)**.** A model category in which the pushout of a weak equivalence along a cofibration is again a weak equivalence, is called left proper.

More concretely, for a weak equivalence $g : A \to B$ and cofibration $i : A \to C$ we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \widetilde{g}} \\ B & \xrightarrow[\ \widetilde{i}\ ]{} & P \end{array}$$

and the requirement is that $\widetilde{g}$ is a weak equivalence. A model category in which all objects are cofibrant, is left proper by Proposition 13.1.2 in [**Hir00**]. So, if the cofibrations are the monomorphisms, then the model category is automatically left proper. This already gives that the category of simplicial sets is left proper.

Another example of a left proper model structure is the projective mode structure. If $\mathscr{M}$ is combinatorial and left proper, then $\mathscr{M}^{\mathbb{D}}$ is left proper. In functor categories pushouts are done pointwise, and weak equivalences are defined pointwise. If we show that every projective cofibration is a pointwise cofibration, then it directly follows that it is left proper. The reason for this is that all generating cofibrations, as defined in Example 3.1.5, are pointwise cofibrations.

Now we can give an existence theorem for left Bousfield S-localizations.

Theorem 3.4.9. *Let $\mathscr{M}$ be a combinatorial left proper model category and let S be a set of maps in $\mathscr{M}$. Then a left Bousfield S-localization of $\mathscr{M}$ exists.*

We will not prove this theorem, and for a proof we refer the reader to [**Hir00**].

CHAPTER 4

Presentations of Model Categories

The main goal of this chapter is to discuss two articles written by Dugger [**Dug01c, Dug01a**], and after that we discuss an application given in [**DHI04**]. The theorems discussed in these articles basically are analogues of several theorems in ordinary category theory and topos theory which tells us something about the structure of certain classes of categories. In ordinary category theory, we have the notion of a locally presentable category, and in Chapter 2 we discussed a theorem which says that all locally presentable categories are reflective subcategories of certain presheaf categories. Also, Giraud's theorem in topos theory tells us that toposes are reflective subcategories of presheaf categories where the left adjoint preserves finite limits. Dugger's theorem is an analogue of these theorems, but it is for combinatorial model categories rather than ordinary categories. To prove and even state this theorem, we need to translate several ordinary concepts into homotopical concepts. For example, in ordinary category theory the presheaf category can be seen as the cocompletion of a small category. To translate this into homotopical language, we need to change several of the concepts involved. Colimits should become homotopy colimits, and sets should become simplicial sets. Also, diagrams will not be required to commute on the nose, but rather up to homotopy. This is basically the idea for translating the theorem and turning it into a theorem about model categories.

The reason why we are interested in this theorem, is because it allows for a nice description of model categories. From this theorem it will follow that every combinatorial model category is equivalent to one that is simplicial, proper, and in which all objects are cofibrant. In simplicial model categories we can compute homotopy colimits using the Bousfield-Kan formula from [**BK87**], so the study of homotopy colimits reduces to the simplicial case. Before this theorem, there was an alternate proof given in [**Dug01b**] that every left proper combinatorial model category is equivalent to a simplicial category. The newer result thus improves on it by weakening the assumptions. Another nice application of this theorem is given in [**DHI04**]. There are several possible model structures for the category of simplicial sheaves, for example the Jardine model structure. Here cofibrations are defined to be the monomorphisms, weak equivalences are defined locally, and the fibrations are defined using lifting properties. The nice thing about this model structure is that all objects are cofibrant. However, the disadvantage is that it is rather difficult to determine the fibrant objects, because the fibrations do not have a nice description. By replacing this model category by a nicer one, the fibrant objects can be described in a nicer way.

4.1. Universal Model Categories

The first part of the result is given in [**Dug01c**], and it says that the category of simplicial presheaves with the projective model structure is the homotopical cocompletion of a small category. The result from ordinary category we would like to translate is the following

Theorem 4.1.1. *Let $\mathscr{C}$ be a small category, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then there is a unique functor $G : \mathrm{Pre}(\mathscr{C}) \to \mathscr{D}$ which preserves all small colimits and such that the following diagram commutes*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ {\scriptstyle y}\downarrow & \nearrow_{G} & \\ \mathrm{Pre}(\mathscr{C}) & & \end{array}$$

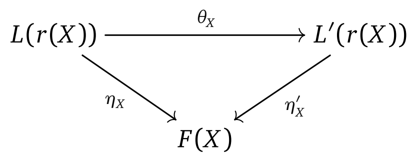
The functor G also has a right adjoint.

For a homotopy theoretic analog we need to translate several things. As given data we will have a small category $\mathscr{C}$ and a functor $F : \mathscr{C} \to \mathscr{M}$ to a model category $\mathscr{M}$. First of all, we need a replacement for $\mathrm{Pre}(\mathscr{C}) = \mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ in the homotopy theoretic world. The category of sets does not really have a nice model structure, and that is why it needs to be replaced. A rule of thumb we use here is that simplicial sets replace sets in homotopy theory, so instead of presheaves we will use simplicial presheaves. The category $\mathbf{SSet}^{\mathscr{C}^{\mathrm{op}}}$ with natural transformations has multiple possible model structures like the projective model structure in Example 3.1.5 or the Reedy model structure from Example 3.1.7, and here we will use the projective model structure. Beside the upcoming theorem, there is another reason why simplicial sets are the homotopic analogue of sets. Presheaves are colimits of representables, and simplicial presheaves are homotopy colimits of representables. So, in homotopy theory simplicial presheaves behave just like presheaves. As expected we have a map $r : \mathscr{C} \to \mathrm{SPre}(\mathscr{C})$ which is given by the composition $\mathscr{C} \to \mathrm{Pre}(\mathscr{C}) \to \mathrm{SPre}(\mathscr{C})$. We can map from $\mathscr{C}$ to $\mathrm{Pre}(\mathscr{C})$ via the Yoneda embedding, and we can map $\mathrm{Pre}(\mathscr{C})$ to $\mathrm{SPre}(\mathscr{C})$ by mapping a presheaf F to the simplicial presheaf $\widetilde{F}$ which is defined in X the constant simplicial set $F(X)$.

Secondly, the commutativity of the diagram will be weakened. Instead of saying that it commutes, we want to say that it commutes up to homotopy. However, this might cause the factorization not to be unique, and that is why we need an extra requirement. This requirement should say that the factorization is unique up to homotopy, and stating this requires some setup. We can form a category where the objects are tuples (L, R, η) where $L \dashv R$ is a Quillen pair with $L : \mathrm{SPre}(\mathscr{C}) \to \mathscr{M} : R$ and η is a natural weak equivalence from $L \circ r$ to F. Such data can be visualized as follows

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{M} \\ {\scriptstyle r}\downarrow & \Downarrow\eta \;\nearrow_{L} & \\ \mathrm{SPre}(\mathscr{C}) & & \end{array}$$

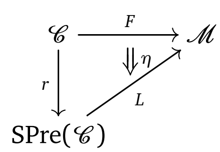
The arrows from (L,R,η) to (L',R',η') are given by natural transformations $\theta : L \Rightarrow L'$ such that the following diagram commutes



This category is called *the category of factorizations* and is denoted by $\mathrm{Fact}_{\mathscr{M}}(\gamma)$.

Lastly, let us recall the notion of a *contractible* category. In Chapter 3 we said that a category $\mathscr{C}$ is contractible iff the realization of its nerve is a contractible topological space. There are many examples of categories which are contractible, and one way to find them, is by finding a terminal object. If a category $\mathscr{C}$ has a terminal object 1, then we have a natural transformation η from the identity functor to the constant functor 1. Recall that Δ^1 is the category with two objects 0 and 1 and one arrow from 0 to 1. Natural transformations $F \Rightarrow G$ where $F, G : \mathscr{C} \to \mathscr{D}$ correspond to natural transformations $\eta : \Delta^1 \times \mathscr{C} \to \mathscr{D}$ such that $\eta(0,C) = F(C)$ and $\eta(1,C) + G(C)$. Now $|\mathscr{N}(\eta)|$ gives a homotopy from the identity, and thus $|\mathscr{N}(\mathscr{C})|$ is contractible. Similarly, if $\mathscr{C}$ has an initial object, then it is contractible as well. Using all this we can formulate the goal of this section

Theorem 4.1.2. *Let $\mathscr{C}$ be a small category, let $\mathscr{M}$ be a model category, and let $F : \mathscr{C} \to \mathscr{M}$. Then there is a Quillen pair $L : \mathrm{Pre}(\mathscr{C}) \to \mathscr{D} : R$ and a natural weak equivalence $\eta : L \circ \gamma \Rightarrow y$ such that the following diagram commutes up to η*



Furthermore, the category $\mathrm{Fact}_{\mathscr{M}}(\gamma)$ *is contractible.*

The main tools to prove this theorem are cosimplicial resolutions. Recall that in Chapter 3 we defined the notion of a cosimplicial resolution in Definition 3.3.2, and using Quillen's small object argument we proved that cosimplicial always exist. The crucial point of the proof is that cosimplicial resolutions of the map $F : \mathscr{C} \to \mathscr{M}$ correspond to factorizations. Then the proof is reduced to working with cosimplicial resolutions, and for them we can prove the desired properties. For example, we can show that a cosimplicial resolution exists, and that the category of cosimplicial resolutions is contractible. From this Theorem 4.1.2 follows directly.

Proposition 4.1.3. *The category of cosimplicial resolutions of γ is contractible.*

The proof of this proposition is rather long and technical, and for a proof we refer the reader to Proposition 16.1.15 in [**Hir00**]. The next proposition says that giving a cosimplicial resolution of γ is the same as factoring γ.

Proposition 4.1.4. *For a diagram $\gamma : \mathscr{C} \to \mathscr{M}$ in a model category $\mathscr{M}$ we have an equivalence of categories*

$$\mathrm{coRes}(\gamma) \simeq \mathrm{Fact}_{\mathscr{M}}(\gamma).$$

PROOF. Recall that in Section 3.3 we defined the operation $\otimes_\Delta$ for cosimplicial resolutions γ and simplicial sets K as follows

$$K \otimes_\Delta \gamma = \operatorname{coeq}\left(\coprod_{[k]\to[m]} K_m \cdot \gamma^k \rightrightarrows \coprod_n K_n \cdot \gamma^n \right).$$

For this operation we have $\mathscr{M}(K \otimes_\Delta \gamma, W) \cong \mathbf{SSet}(K, \mathscr{M}(\gamma^*, W))$ where $\mathscr{M}(\gamma^*, W))$ is the simplicial set which is $\mathscr{M}(\gamma^n, W))$ in degree n. The reason for this is as follows. A map $K \to \mathscr{M}(\gamma^*, W)$ of simplicial sets corresponds with maps $K_n \to \mathscr{M}(\gamma^n, W)$ which make certain diagrams commute. But a map $K_n \to \mathscr{M}(X_n, W)$ corresponds with a map $K_n \cdot X_n \to W$ in $\mathscr{M}$, because every element in K gets mapped to a map $X_n \to W$. We know more, namely that the map $K \to \mathscr{M}(\gamma^*, W)$ is a map of simplicial sets which says that for any arrow $[m] \to [n]$ the following diagram commutes

$$\begin{array}{ccc} K_m & \longrightarrow & \mathscr{M}(\gamma^m, W) \\ \downarrow & & \downarrow \\ K_n & \longrightarrow & \mathscr{M}(\gamma^n, W) \end{array}$$

This diagram gives another diagram, which should commute as well, namely

$$\begin{array}{ccc} K_m \cdot X_m & & \\ \uparrow & \searrow & \\ K_m \cdot X_n & \longrightarrow & W \\ \downarrow & \nearrow & \\ K_n \cdot X_n & & \end{array}$$

These arrows are precisely the arrows in the coequalizer of $K \otimes_\Delta \gamma$, and that is why a map in $\mathbf{SSet}(K, \mathscr{M}(\gamma^*, W))$ corresponds to a map $\mathscr{M}(\gamma \otimes_\Delta K, W)$.

The next notation we introduce for this proof is $\otimes_\mathscr{C}$ which is defined for diagrams $\Gamma : \mathscr{C} \to c\mathscr{M}$ and simplicial presheaves $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{SSet}$ as follows

$$F \otimes_\mathscr{C} \Gamma = \operatorname{coeq}\left(\coprod_{a\to b} F(b) \otimes_\Delta \Gamma(a) \rightrightarrows \coprod_c F(c) \otimes_\Delta \Gamma(c) \right).$$

Here the first coproduct is over all arrows $a \to b$ in $\mathscr{C}$ and the second coproduct is over all objects in $\mathscr{C}$. Note the similarity between this formula and the formula for $\otimes_\Delta$, and in a similar way we can show that $\mathscr{M}(F \otimes_\mathscr{C} \Gamma, W) \cong \mathbf{SSet}^{\mathscr{C}^{\mathrm{op}}}(F, \mathscr{M}(\Gamma, W))$ where we define $\mathscr{M}(\Gamma, W)$ to be the presheaf $c \mapsto \mathscr{M}(\Gamma(c), W)$.

Let us denote the representable functors of $\mathrm{Pre}(\mathscr{C} \times \Delta)$ by $r_{X,n}$, and note that $r(X) = r_{X,0}$ For $\otimes_\mathscr{C}$ we have a property similar to Proposition 3.3.4 which says that $r_{X,n} \otimes_\mathscr{C} \Gamma \cong \Gamma(X)^n$. The proof of these two are similar, because it only depend on the definition of the representables. Now we have the required notation to define the functors. Let us start with a factorization $L \dashv R$ with a natural weak equivalence $\eta : L(r(X)) \to \gamma(X)$. Then we define $G((L, R, \eta))^n = L(r_{X,n})$. For a cosimplicial resolution Γ we need to define $H(\Gamma) = (L, R, \eta)$. We define $L(F) = F \otimes_\mathscr{C} \Gamma$ and $R(X)(c) = \mathscr{M}(\Gamma^*(c), X)$. Note that $L \dashv R$, because $\mathscr{M}(F \otimes_\mathscr{C} \Gamma, W) \cong \mathbf{SSet}^{\mathscr{C}^{\mathrm{op}}}(F, \mathscr{M}(\Gamma, W))$ and thus this is an adjunction. The only thing missing now is a natural weak equivalence $L(r(X)) \to \gamma(X)$. This can be found using the fact that

$$L(r(X)) = r(X) \otimes_{\mathscr{C}} \Gamma = r_{X,0} \otimes_{\mathscr{C}} \Gamma \cong \Gamma(X)^0,$$

and the fact we have a weak equivalence $\Gamma(X)^0 \to \gamma(X)$.

To check that this is an equivalence, we need to check two things. First of all, we need to check that $\Gamma(X)^n$ is naturally isomorphic to $r_{X,n} \otimes_{\mathscr{C}} \Gamma$. This is the case, as we said earlier. For the other check we write $\Gamma(X)^n = L(r_{X,n})$, and now we need to check that $L(F) \cong F \otimes_{\mathscr{C}} \Gamma$. Let $\mathscr{C} \times \Delta \downarrow F$ be the category whose objects are tuples $(n, X, r_{X,n} \to F)$, and let define a functor I which sends such a tuple to $r_{X,n} \to F$. The colimit of I is equal to F, because presheaves are colimits of representables. Write Γ' for the cosimplicial object $\Gamma'(X)^n = r_{X,n}$, and now we have

$$\begin{aligned} F &= \operatorname{colim}_I(r_{X,n}) \\ &= \operatorname{colim}_I \Gamma'(X)^n \\ &= \operatorname{colim}_I(r_{X,n} \otimes_{\mathscr{C}} \Gamma') \\ &= (\operatorname{colim}_I r_{X,n}) \otimes_{\mathscr{C}} \Gamma' \\ &= F \otimes_{\mathscr{C}} \Gamma' \end{aligned}$$

Because L is a left adjoint, it commutes with colimits, so we get $L(F) = L(F \otimes_{\mathscr{C}} \gamma') = F \otimes_{\mathscr{C}} L(\Gamma') = F \otimes_{\mathscr{C}} \Gamma$. □

The next property we need is an analogue of the categorical fact that every presheaf is a colimit of representables. This is a bit more complicated for simplicial presheaves, because they have multiple degrees.

Proposition 4.1.5. *Let F be a simplicial presheaf on $\mathscr{C}$. Define a functor $L : \mathscr{C} \times \Delta \to$* Pre$(\mathscr{C} \times \Delta)$ *sending (C, n) to $r_{C,n}$, and write the homotopy colimit of this functor as* hocolim$(\mathscr{C} \times \Delta \downarrow F)$. *Then the natural arrow* hocolim$(\mathscr{C} \times \Delta \downarrow F) \to F$ *is a weak equivalence.*

We will not prove this in detail, but rather give a sketch. The first main point is to reduce it to simplicial sets, and that can be done, because weak equivalences are defined pointwise and homotopy colimits are computed pointwise. Therefore, it is sufficient to show that the arrow hocolim$(r_{X,n}(C)) \to F(X)$ is a weak equivalence.

For this we define two categories. First, we define a category $\Delta(X, F)$ whose objects are pairs $([n], r_{X,n} \to F)$, and we define a functor $G : \Delta(X, F) \to$ **SSet** sending $([n], r_{X,n} \to F)$ to $\Delta[n]$. One can show that the colimit of this diagram is $F(X)$, and that the map $\operatorname{hocolim}_{\Delta(X,F)} G \to F(X)$ is a weak equivalence.

Let I be the category with objects $(C, [n], y_{C,n} \to F)$. Next we define $\Theta : I \to$ **Sets** to be the functor which maps $(C, [n], y_{C,n} \to F)$ to $\mathscr{C}(X, C)$. For this functor we can consider the Grothendieck construction Gr(Θ). We define Gr(θ) to be the category with objects (i, σ) where i is an object of I and $\sigma \in \Theta(i)$. Arrows from (i, σ) to (j, τ) are maps $f : i \to j$ such that $\Theta(f)(\sigma) = \tau$. Now let H : Gr$(\theta) \to$ **SSet** be the functor sending $(C, [n], y_{C,n} \to F)$ to $\Delta[n]$. By Corollary 24.6 from [**CS01**] we have a weak equivalence $\operatorname{hocolim}_{\mathrm{Gr}(\Theta)} \to \operatorname{hocolim}(r_{X,n}(C))$.

Lastly, we can define a functor $\Delta(X, F) \to \mathrm{Gr}(\Theta)$ sending $([n], r_{X,n} \to F)$ to (i, σ) with $i = (X, [n], r_{X,n} \to F)$ and $\sigma = \mathrm{Id} : X \to$. This gives a map $\operatorname{hocolim}_{\Delta(X,F)} G \to \operatorname{hocolim}_{\mathrm{Gr}(\theta)} H$ which has a retraction. One can show that its retraction is cofinal, and therefore the map $\operatorname{hocolim}_{\Delta(X,F)} G \to \operatorname{hocolim}_{\mathrm{Gr}(\theta)} H$ is a weak equivalence.

All in all, we get the following diagram

$$\begin{array}{ccccc} \operatorname{hocolim}_{\Delta(X,F)} G & \longrightarrow & \operatorname{hocolim}_{\mathrm{Gr}(\theta)} H & \longrightarrow & \operatorname{hocolim}(r_{X,n}(C)) \\ & \searrow & & & \downarrow \\ & & & & F(X) \end{array}$$

As said before, all arrows but the vertical one are weak equivalences, and thus by the 2-out-of-3 property we can conclude that the arrow $\operatorname{hocolim}(r_{X,n}(C)) \to F(X)$ is a weak equivalence as well.

4.2. Presentations of Model Categories

Using the universal model category we formulate and prove the required analogue of Giraud's theorem

Theorem 4.2.1. *Let $\mathscr{M}$ be a combinatorial model category. Then there is a small category $\mathscr{C}$ and a set of maps S in $\mathrm{SPre}(\mathscr{C})$ such that the induced map $L : \mathrm{SPre}(\mathscr{C}) \to \mathscr{M}$ sends maps in S to weak equivalences in $\mathscr{M}$ and $\mathrm{SPre}(\mathscr{C})/S \to \mathscr{M}$ is a Quillen equivalence.*

The localization $\mathrm{SPre}(\mathscr{C})/S$ is a Bousfield localization and it exists, because the model category of simplicial presheaves is proper and combinatorial.

The proof of Theorem 4.2.1 is done in two steps, and it mimics the theorem that every abelian group has a free presentation. Recall that to prove that an abelian group has a free presentation, we first find the generators and then we find the relations. For model categories we do the same things. First, we find the generators, so we look for a small category $\mathscr{C}$ and a 'surjective' map $\mathrm{SPre}(\mathscr{C}) \to \mathscr{M}$. Here we need a special notion of surjectivity which will be defined in Definition 4.2.2. The next step will thus be to find relations, so we look for a set S such that $\mathscr{M}$ is Quillen-equivalent to $\mathrm{SPre}(\mathscr{C})/S$. Let us start with giving the right notion of surjectivity for this proof.

Definition 4.2.2 (Homotopically Surjective)**.** Let $L : \mathscr{M} \to \mathscr{N}$ be a left Quillen functor with right adjoint R.. Then we say L is *homotpically surjective* iff for all fibrant objects X and cofibrant replacements $Y \xrightarrow{\sim} R(X)$ the composition map $L(Y) \to L(R(X)) \to X$ is a weak equivalence.

This says that the left derived functor of L has a section, namely R. The idea is that we do not want to say that L is surjective, but rather that it is surjective in the homotopy category. Hence, we have to look at derived functors, and this definition says that $\mathbb{L}L$ is surjective.

For the proof we need two steps which are given in the following propositions, and the proof of them will require the rest of this section.

Proposition 4.2.3. *Let $\mathscr{M}$ be a combinatorial model category. Then there is a small category $\mathscr{C}$ and a homotopically surjective map $\mathrm{SPre}(\mathscr{C}) \to \mathscr{M}$.*

Proposition 4.2.4. *Let $\mathscr{M}$ and $\mathscr{N}$ be combinatorial model categories such that $\mathscr{M}$ is left proper, and suppose that we have a functor $L : \mathscr{M} \to \mathscr{N}$ which is homotopically surjective. Then there is a set S of maps which becomes weak equivalences under $\mathbb{L}L$ such that $\mathscr{M}/S \to \mathscr{N}$ is a Quillen equivalence.*

If we can prove these two propositions, then Theorem 4.2.1 follows immediately. Let us start with the second proposition, because it is easier. To prove it, we need one technical lemma.

Lemma 4.2.5. *Let $\mathscr{M}$ be a combinatorial model category. Then there is a cardinal number λ such that*

- (i) *There are fibrant and cofibrant replacement functors which preserve λ-filtered colimits.*
- (ii) *λ-filtered colimits of weak equivalences are again weak equivalences.*
- (iii) *The cofibrant and fibrant replacement of λ-small objects is again λ-small.*

PROOF. The first follows by closely inspecting Quillen's small object argument. The main point is that colimits commute with colimits, but there is one subtlety. If α is an ordinal, and we would like to construct $X_{\alpha+1}$, then we looked at all diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

with A and B are λ-small. If X_α and Y can be written as a λ-filtered colimit, say $X_\alpha = \operatorname{colim}_\beta X'(\beta)$ and $Y = \operatorname{colim}_\beta Y'(\beta)$, then this dagram can be written as the colimit of the diagrams

$$\begin{array}{ccc} A & \longrightarrow & X'_\beta \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y'_\beta \end{array}$$

Therefore, the constructed functor commutes with λ-filtered colimits.

Next we show that the λ-filtered colimit of weak equivalences is again a weak equivalence for λ sufficiently large. Let λ be an ordinal such that the functorial factorizations preserve λ-filtered colimits and $\mathscr{M}$ is generated by λ-small objects. Suppose we have two diagrams $D_1, D_2 : I \to \mathscr{M}$ with I a λ-filtered category, and a natural transformation $\eta : D_1 \Rightarrow D_2$ where each η_i is a weak equivalence. For each object i of I we can factorize η_i as follows

$$\begin{array}{ccccc} D_1(i) & & \xrightarrow{\quad \eta_i \quad} & & D_2(i) \\ & {\scriptstyle f_i}\searrow & & \nearrow{\scriptstyle p_i} & \\ & & \widetilde{D}(i) & & \end{array}$$

with f_i a trivial cofibration and p_i a fibration. For $\operatorname{colim} D_1 \to \operatorname{colim} D_2$ we can get a factorization into a trivial cofibration followed by a fibration by taking the λ-filtered colimit of the f_i, so the map $\operatorname{colim} f_i$ is a trivial cofibration. Since the functorial factorizations preserve λ-filtered colimits by the choice of λ, we thus get the following factorization

$$\begin{array}{ccccc} \operatorname{colim} D_1 & & \xrightarrow{\quad \operatorname{colim} \eta_i \quad} & & \operatorname{colim} D_2 \\ & {\scriptstyle \operatorname{colim} f_i}\searrow & & \nearrow{\scriptstyle \operatorname{colim} p_i} & \\ & & \operatorname{colim} \widetilde{D} & & \end{array}$$

This factorizes the map $\operatorname{colim}\eta_i$ in a trivial cofibration followed by a fibration. Thus, to show $\operatorname{colim}\eta_i$ is a weak equivalence, it suffices to show that $\operatorname{colim} p_i$ is a weak equivalence. Each p_i is a weak equivalence by the 2-out-of-3 property, and thus it suffices to show that the λ-filtered colimit of trivial fibrations is again a trivial fibration.

To check whether a map is a trivial fibration, we can check that it satisfies the right lifting property with respect to the generating cofibrations. Let $g : A \to B$ be a generating cofibration between λ-small objects. Consider the following diagram

$$\begin{array}{ccc} A & \longrightarrow & \operatorname{colim} X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \operatorname{colim} p_i} \\ B & \longrightarrow & \operatorname{colim} Y \end{array}$$

Since A is λ-small, the map $A \to \operatorname{colim} X$ factors through some X_i. Also, B is λ-small as well, and thus the $B \to \operatorname{colim} Y$ factors through some Y_j. Taking m to be the maximum of i and j, the map $A \to \operatorname{colim} X$ factors through X_m and the map $B \to \operatorname{colim} Y$ factors through Y_m. Now we have the following diagram

$$\begin{array}{ccccc} A & \longrightarrow & X_m & \longrightarrow & \operatorname{colim} X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle \operatorname{colim} f_i} \\ B & \longrightarrow & Y_m & \longrightarrow & \operatorname{colim} Y \end{array}$$

We can find a lift $B \to X_m$, because f_m is a trivial fibration and $A \to B$ is a cofibration. This gives a lift for the original lifting problem, and from this we can conclude that the λ-filtered colimit of trivial fibrations is again a trivial fibration. All in all, the λ-filtered colimit of weak equivalences is again a weak equivalence.

Lastly, we need to show that the cofibrant replacement and the fibrant replacement of a λ-small object is again λ-small. We shall show that for the cofibrant replacement, because for the fibrant replacement a similar argument can be given. Pick an ordinal λ such that $\mathscr{M}$ is generated by a set S of λ-small objects and such that the functorial factorization preserves λ-filtered colimits. Note that there is only a set of maps between objects of S, because $\mathscr{M}$ is locally small. We can factorize all these maps using the functorial factorizations, and this gives a set T of objects. Every object in T is small, and thus there is a an ordinal ν such that the factorization of a map between objects in S gives a ν-small object.

Let μ be larger than both λ and ν. We claim that μ is the required ordinal, and for that we take a map $A \to B$ between μ-small objects. By local presentability we can write $A = \operatorname{colim} A_\alpha$ and $B = \operatorname{colim} B_\beta$ where the colimits are λ-filtered and each A_α and B_β is λ-small. A map $X \to Y$ thus correspond with maps $X_\alpha \to Y$ for every α, and using the smallness of X_α this corresponds with maps $X_\alpha \to Y_{\beta(\alpha)}$. Hence, we can write $X \to Y$ as a λ-filtered colimit of maps $X_\alpha \to Y_\alpha$ between λ-small objects.

To factorize $X \to Y$, we factorize $X_\alpha \to Y_\alpha$ as $X_\alpha \to \widetilde{X}_\alpha \to Y_\alpha$. By the construction of ν, the objects $\widetilde{X}_\alpha$ are ν-small. Now we can factorize $X \to Y$ as $X \to \operatorname{colim} \widetilde{X}_\alpha \to Y$, and note that $\operatorname{colim} \widetilde{X}_\alpha$ is a λ-filtered colimit of ν-small objects where the diagram has size μ. Since μ is both larger than ν and λ, we get that it is a μ-filtered colimit of μ-small objects on a diagram of size μ. Now it follows from Proposition 1.16 from [**AR94**] that it is a μ-small object. □

Now we can prove Proposition 4.2.4 with this lemma, and before we give the precise proof, we give a sketch. The main idea of this proof is that we want to turn the functor into a Quillen equivalence, and that means that certain maps have to be weak equivalences. By Proposition 2.1.3 it is sufficient to check that the maps $X \xrightarrow{\eta_X} R(L(X)) \longrightarrow R([L(X)]^{\mathrm{Fib}})$ and $L([R(Y)]^{\mathrm{Cof}}) \longrightarrow L(R(Y)) \xrightarrow{\varepsilon_Y} Y$ are weak equivalences for all cofibrant X and fibrant Y. Because L is homotopically surjective, the maps $L(Y) \to L(R(X)) \to X$ are weak equivalences where Y is a cofibrant replacement of $R(X)$. We want to localize the maps $X \xrightarrow{\eta_X} R(L(X)) \longrightarrow R([L(X)]^{\mathrm{Fib}})$, but these maps might not form a set. Hence, we look at a set of small objects which generate the model category. Then only some minor technical problems are left which can be solved by choosing λ big enough.

PROOF OF PROPOSITION 4.2.4. We start by choosing λ big enough such that

(1) $\mathscr{M}$ is generated by a set S of λ-small objects.
(2) λ-filtered colimits of weak equivalences are again weak equivalences.
(3) The cofibrant and fibrant replacement functors preserve λ-filtered colimits.
(4) The right adjoint R preserves λ-filtered colimits.

For (1) we use the assumption that $\mathscr{M}$ is combinatorial. For (2) and (3) we use Lemma 4.2.5, and for the last we use Proposition 1.66 from [**AR94**]. The set to which we localize is then defined as

$$T = \{A^{\mathrm{Cof}} \to R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}}) \mid A \in S\}$$

To finish the proof, we need to check some things. First of all, we need to check that all maps in T are send to weak equivalences by the left derived functor of L. This will give a functor $\widetilde{L} : \mathscr{M}/S \to \mathscr{N}$, and we claim $\widetilde{L}$ is a Quillen equivalence. For that we use Proposition 2.1.3 which will follow from the assumption that L is homotopically surjective and from the definition of T.

Now let us do all those checks. First we check that for $A \in S$ the map $L(A^{\mathrm{Cof}}) \to L([R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}})]^{\mathrm{Cof}})$ is a weak equivalence. For that we look at the diagram

$$\begin{array}{ccccc} [L(A^{\mathrm{Cof}})]^{\mathrm{Fib}} & \longrightarrow & L([R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}})]^{\mathrm{Cof}}) & \longrightarrow & L(R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}})) \\ & \searrow & \downarrow \sim & \swarrow_{\varepsilon} & \\ & & [L(A^{\mathrm{Cof}})]^{\mathrm{Fib}} & & \end{array}$$

The arrow $L([R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}})]^{\mathrm{Cof}}) \to L(A^{\mathrm{Cof}})$ is a weak equivalence, because L is homotopically surjective. Also, if we denote the weak equivalence $L(A^{\mathrm{Cof}}) \to [L(A^{\mathrm{Cof}})]^{\mathrm{Fib}}$ as i, then the arrow $L(A^{\mathrm{Cof}}) \to L(R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}}))$ is precisely $L(R(i) \circ \eta)$, so the composition $L(A^{\mathrm{Cof}}) \to [L(A^{\mathrm{Cof}})]^{\mathrm{Fib}}$ is the weak equivalence $\varepsilon \circ L(R(i) \circ \eta) = i$. Hence, by the 2-out-of-3 property the map $[L(A^{\mathrm{Cof}})]^{\mathrm{Fib}} \to L([R([L(A^{\mathrm{Cof}})]^{\mathrm{Fib}})]^{\mathrm{Cof}})$ is indeed a weak equivalence.

Note that $\widetilde{L}$ is homotopically surjective. Since the Bousfield localization $\mathscr{M}/T$ has the same objects as $\mathscr{M}$, we need to check that for every fibrant object X and cofibrant replacements $Y \xrightarrow{\sim} R(X)$ the composition map $L(Y) \to L(R(X)) \to X$ is a weak equivalence. But weak equivalences in $\mathscr{M}$ are weak equivalences in $\mathscr{M}/T$, so all these maps are indeed weak equivalences.

Lastly, we check that $\widetilde{L}$ is a Quillen equivalence. Let X be cofibrant and let Y be fibrant. Because $\widetilde{L}$ is homotopically surjective, the map $\widetilde{L}(([\widetilde{R}(Y))]^{\mathrm{Cof}}) \to Y$ is a weak equivalence. Next we check that the $X \to R([L(X)]^{\mathrm{Fib}})$ are weak equivalences, and for that we write X as $\mathrm{colim}_I A_i$ where I is λ filtered. All maps $A_i^{\mathrm{Cof}} \to R([L(A_i^{\mathrm{Cof}})]^{\mathrm{Fib}})$ are weak equivalences, and all involved functors commute with λ-filtered colimits. Since weak equivalences are closed under λ-filtered colimits, the map $X^{\mathrm{Cof}} \to R([L(X^{\mathrm{Cof}})]^{\mathrm{Fib}})$ is a weak equivalence. Hence, $\widetilde{L}$ is indeed a Quillen equivalence and this proves the proposition. □

Next we shall prove the more difficult proposition which finds the generators. The important thing here is that tthe category of simplicial presheaves on $\mathscr{C}$ can be seen as the free homotopy colimit completion of $\mathscr{C}$, so every simplicial presheaf is a homotopy colimit of $r(C)$ with C an object of $\mathscr{C}$. So, if we find a small category $\mathscr{C}$ and a map $F : \mathscr{C} \to \mathscr{M}$ such that we can write every object in $\mathscr{M}$ as homotopy colimit of the $F(C)$, then we would expect that $\mathscr{M}$ is equivalent to $\mathrm{SPre}(\mathscr{C})$ with possibly some extra relations. More precisely, note that the inclusion $\mathscr{C} \to \mathscr{M}$ gives a map $\mathrm{SPre}(\mathscr{C}) \to \mathscr{M}$, and by the previous argument we expect that the latter map would be homotopically surjective.

Since $\mathscr{M}$ is locally presentable, we have an obvious choice of the generators. By definition every object of $\mathscr{M}$ there is a set of λ-small objects such that every object of $\mathscr{M}$ can be written as a colimit of these λ-small objects. However, this does not turn out to be sufficient, and the problem is that it ignores higher homotopies. The solution is to add these higher homotopies by taking a cosimplicial resolution, and we take the generators to be everything occurring in this cosimplicial resolution. Hence, our generators are not just the λ-small objects, but also their higher homotopies.

Now there is a minor problem: how do we recognize whether everything can be written as a homotopy colimit of the generators? For that we take inspiration from the proof that every presheaf is a colimit of representables. When proving that every presheaf is the colimit of representables, we use the Yoneda embedding $\mathscr{C} \to \mathrm{Pre}(\mathscr{C})$. To write an arbitrary presheaf F as a colimit of the $y(C)$, we look at a certain overcategory, namely $\mathscr{C} \downarrow F$. The objects of this category are natural transformations $y(C) \to F$, and the arrows are commutative diagrams of the following form

$$\begin{array}{ccc} y(C) & \xrightarrow{y(f)} & y(C') \\ & \searrow \quad \swarrow & \\ & F & \end{array}$$

where f is an arrow from C' to C. This can be seen as a canonical way of writing F as a colimit of representables, because if it would be possible, then this colimit would work. Translating this into homotopy theorem gives the following definition

Definition 4.2.6. Let $\mathscr{C}$ be a category, and let a functor $\gamma : \mathscr{C} \to \mathscr{M}$ into a model category be given. Given an object X of $\mathscr{M}$, define a category $\mathscr{C} \downarrow X$ where objects are arrows $\gamma(C) \to X$ and arrows commutative diagrams. We write $\mathrm{hocolim}(\mathscr{C} \downarrow X)$ for the homotopy colimit of this diagram.

This definition is problematic in a certain way. As said before the generators will be everything in the cosimplicial resolution of the γ, but this colimit does not take this cosimplicial resolution into consideration. To correct this, we introduce the notion of a *canonical homotopy colimit* which is almost the same.

Definition 4.2.7 (Canonical Homotopy Colimit)**.** Let $\mathscr{C}$ be a category, let a functor $\gamma : \mathscr{C} \to \mathscr{M}$ into a model category be given, and let $\Gamma : \mathscr{C} \to \mathrm{c}\mathscr{M}$ be a cosimplicial resolution of γ. Given a fibrant object X of $\mathscr{M}$, define a category $\mathscr{C} \times \Delta \downarrow X$

where objects are arrows $\Gamma^n(C) \to X$ and arrows commutative diagrams. We write $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ for the homotopy colimit of this diagram, and it is called the *canonical homotopy colimit*.

It is not obvious to prove that this is well-defined, so that it does not depend on the chosen cosimplicial resolution. To prove this, we will need the assumption that X is fibrant. Note that for each cosimplicial resolution Γ we can define $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ as in Definition 3.3.5.

Beside the fact that the canonical homotopy colimit takes the cosimplicial resolution into consideration, there is another reason why this notion is important. Namely, we can recognize with it whether maps $\mathrm{SPre}(\mathscr{C}) \to \mathscr{M}$ are homotopically surjective. The statement says that such a map is homotopically surjective precisely when all canonical homotopy colimits $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ are weakly equivalent to X. This follows from the following proposition

Proposition 4.2.8. *Let $L : \mathrm{SPre}(\mathscr{C}) \leftrightarrow \mathscr{M} : R$ be induced by Γ. Then the object $L([R(X)]^{Cof})$ and $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ are weakly equivalent.*

PROOF. We have shown in the previous section that $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow F) \to F$ is a cofibrant approximation of the presheaf F. In particular, $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow R(X)) \to R(X)$ is a cofibrant approximation of $R(X)$. To compute $L([R(X)]^{\mathrm{Cof}})$, we use this formula, and this gives that $L([R(X)]^{\mathrm{Cof}})$ and $L(\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow R(X)))$ are weakly equivalent.

Let us briefly recall the objects of the involved categories. The objects of $\mathscr{C} \times \Delta \downarrow R(X)$ are arrows $r_{C,n} \to R(X)$ and the objects of $\mathscr{C} \times \Delta \downarrow X$ are arrows $\Gamma^n(C) \to X$. For both categories the arrows are commutative triangles in the obvious way. By adjunction arrows $r_{C,n} \to R(X)$ correspond with arrows $L(r_{C,n}) \to X$, and remember that we saw that $L(r_{C,n})$ and $\Gamma^n(C)$ are isomorphic in the proof of Proposition 4.1.4. Hence, the categories $\mathscr{C} \times \Delta \downarrow R(X)$ and $\mathscr{C} \times \Delta \downarrow X$ are isomorphic.

Since L is a left adjoint, it preserves colimits and thus it preserves homotopy colimits as well. For $L(\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow R(X)))$ we get a homotopy colimit over a category isomorphic to $\mathscr{C} \times \Delta \downarrow X$ and we get a pointwise weak equivalence. Hence, $L(\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow R(X)))$ and $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ are weakly equivalent which proves the proposition. □

Note that we have the following diagram now

$$\begin{array}{ccc} \mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X) & \xrightarrow{\sim} & L(R(X)^{\mathrm{Cof}}) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

From the 2-out-of-3 property follows that $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X) \to X$ is a weak equivalence iff $L(R(X)^{\mathrm{Cof}}) \to X$ is a weak equivalence. The second of these two just says that L is homotopically surjective, and thus we get

Corollary 4.2.9. *Let $\gamma : \mathscr{C} \to \mathscr{M}$ be a functor and let $\Gamma : \mathscr{C} \to \mathrm{c}\mathscr{M}$ be a cosimplicial resolution of γ. Then the induced map $\mathrm{SPre}(\mathscr{C}) \to \mathscr{M}$ is homotopically surjective iff $\mathrm{hocolim}(\mathscr{C} \times \Delta \downarrow X) \to X$ is a weak equivalence for all fibrant objects X.*

From this we can conclude that the canonical homotopy colimit is indeed well-defined.

Corollary 4.2.10. *If we have a weak equivalence $X \to Y$ with both X and Y fibrant, then* $\text{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ *and* $\text{hocolim}(\mathscr{C} \times \Delta \downarrow Y)$ *are weakly equivalent.*

PROOF. We showed that $\text{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ and $L([R(X)]^{\text{Cof}})$ are weakly equivalent, and that $\text{hocolim}(\mathscr{C} \times \Delta \downarrow Y)$ and $L([R(Y)]^{\text{Cof}})$ are weakly equivalent. Recall that by Proposition 2.1.8 left Quillen functors preserve weak equivalences between cofibrant objects, and that we can show similarly that R preserves weak equivalences between fibrant objects. Since X and Y are fibrant and we have a weak equivalence $X \to Y$, we get a weak equivalence $R(X) \to R(Y)$. Then we also get a weak equivalence $R(X)^{\text{Cof}} \to R(Y)^{\text{Cof}}$, and since both objects are cofibrant, we get a weak equivalence $L([R(X)]^{\text{Cof}}) \to L([R(Y)]^{\text{Cof}})$. □

Corollary 4.2.11. *Let Γ and Γ' be two cosimplicial resolutions for γ. Note that Γ induces an adjunction $L \dashv R$ and that Γ' gives an adjunction $L' \dashv R'$. Then $L([R(X)]^{Cof})$ and $L'([R'(X)]^{Cof})$ are weakly equivalent for fibrant objects X meaning that there is a zig-zag of weak equivalences between them.*

PROOF. Since the category of cosimplicial resolutions is contractible by Proposition 4.1.3, we can find a zig zag $\Gamma_0, \dots, \Gamma_n$ of maps between $\Gamma = \Gamma_0$ and $\Gamma' = \Gamma_n$. Suppose that $f_i : \Gamma_i \to \Gamma_{i+1}$, and then we can look at the diagram

$$\begin{array}{ccc} \Gamma_i & \longrightarrow & \Gamma_{i+1} \\ & \searrow^{\sim} \quad {}^{\sim}\swarrow & \\ & c^*(\gamma(-)) & \end{array}$$

By the 2-out-of-3 property we can now conclude that the map $\Gamma_i \to \Gamma_{i+1}$ is a weak equivalence, and thus without loss of generality we can assume that we have a weak equivalence $\Gamma \to \Gamma'$.

Recall that by the proof of Proposition 4.1.4 we have $L(F) = F \otimes_{\mathscr{C}} \Gamma$ and that $L'(F) = F \otimes_{\mathscr{C}} \Gamma'$. Also, from the same proof we can conclude that $R(X)(c) = \mathscr{M}(\Gamma^*(c), X)$ and that $R'(X)(c) = \mathscr{M}((\Gamma')^*(c), X)$. From Corollary 16.5.5 in [**Hir00**] follows that $R(X)$ and $R'(X)$ are weakly equivalent for fibrant X, and from Corollary in [**Hir00**] it follows that $L(F)$ and $L'(F)$ are weakly equivalent for cofibrant F. Concluding, all the maps in the following diagram are weak equivalences

$$\begin{array}{ccc} L([R'(X)]^{\text{Cof}}) & \longrightarrow & L([R(X)]^{\text{Cof}}) \\ \downarrow & & \downarrow \\ L'([R'(X)]^{\text{Cof}}) & \longrightarrow & L'([R(X)]^{\text{Cof}}) \end{array}$$

Hence, $L([R(X)]^{\text{Cof}})$ and $L'([R'(X)]^{\text{Cof}})$ are weakly equivalent □

Combining Proposition 4.2.8 and corollary 4.2.11 we can conclude that the canonical homotopy colimit is well-defined for fibrant objects X. However, the disadvantage of the canonical homotopy colimit is that we can work more easily with $\text{hocolim}(\mathscr{C} \downarrow X)$, because it does not involve a cosimplicial resolution of γ. Therefore, we would like to know when these two are equivalent, and this answered by the following proposition. Basically, it says that if all the higher homotopies are equivalent to the original, then $\text{hocolim}(\mathscr{C} \downarrow X)$ and $\text{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ agree.

Proposition 4.2.12. *Let $\gamma : \mathscr{C} \to \mathscr{M}$ be a functor which maps into the cofibrant objects of $\mathscr{M}$. Suppose that X is a fibrant object, and write $\mathscr{C}^n \downarrow X$ for the category where the objects are $\Gamma^n(C) \to X$ and the arrows are commutative triangles. If* $\text{hocolim}(\mathscr{C}^0 \downarrow X) \to \text{hocolim}(\mathscr{C}^n \downarrow X)$ *is a weak equivalence for all n, then also the map* $\text{hocolim}(\mathscr{C} \downarrow X) \to \text{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ *is a weak equivalence.*

The proof of this proposition is rather complicated, and for its proof we refer the reader to [**Dug01a**]. This gives us a method to compare these two homotopy colimits, and this will be very useful. It is much simpler to recognize whether $\text{hocolim}(\mathscr{C} \downarrow X)$ is equivalent to something than to recognize whether $\text{hocolim}(\mathscr{C} \times \Delta \downarrow X)$ is equivalent to it.

Since combinatorial model categories are locally presentable, they have generators. In a local presentable category $\mathscr{C}$ every object X can be written as a λ-filtered colimit of the generators, namely we can write it as $\text{colim}(\mathscr{C}_\lambda \downarrow X)$ where $\mathscr{C}$ is the full subcategory of λ-small objects of $\mathscr{C}$. The following proposition is a direct analogue of this result.

Proposition 4.2.13. *Let $\mathscr{M}$ be a combinatorial model category. Then there is a cardinal number λ such that for all objects X*

- $\text{hocolim}(\mathscr{M}_\lambda \downarrow X) \to X$ *is a weak equivalence;*
- $\text{hocolim}(\mathscr{M}_\lambda^{\text{cof}} \downarrow X) \to X$ *is a weak equivalence.*

PROOF. To prove that $\text{hocolim}(\mathscr{M}_\lambda \downarrow X) \to X$ is a weak equivalence, we note that we have a natural maps $\text{hocolim}(\mathscr{M}_\lambda \downarrow X) \to \text{colim}(\mathscr{M}_\lambda \downarrow X) \to X$. Since $\mathscr{M}$ is locally presentable, we can find a λ such that $\text{colim}(\mathscr{M}_\lambda \downarrow X) \to X$ is an isomorphism. The map $\text{hocolim}(\mathscr{M}_\lambda \downarrow X) \to \text{colim}(\mathscr{M}_\lambda \downarrow X)$ is a λ-filtered colimit of weak equivalences, because the homotopy colimit is the left derived functor of the colimit. By Lemma 4.2.5 we can take λ such that λ-filtered colimits of weak equivalences are again weak equivalences, and this makes the map $\text{hocolim}(\mathscr{M}_\lambda \downarrow X) \to \text{colim}(\mathscr{M} \downarrow X)$ a weak equivalence. Hence, the composition $\text{hocolim}(\mathscr{M}_\lambda \downarrow X) \to X$ is a weak equivalence too.

To check the second statement, we apply Proposition 3.3.6. We have to show that $\text{hocolim}(\mathscr{M}_\lambda)$ and $\text{hocolim}(\mathscr{M}_\lambda^{Cof})$ are equivalent, and for that we first need to make an equivalence between the indexing categories. By Lemma 4.2.5 we have a cofibrant replacement functor which preserves λ-small objects, and this gives a functor $F : \text{hocolim}(\mathscr{M}_\lambda) \to \text{hocolim}(\mathscr{M}_\lambda^{Cof})$. For the functor in the other direction we can take the inclusion $G : \text{hocolim}(\mathscr{M}_\lambda^{Cof}) \to \text{hocolim}(\mathscr{M}_\lambda)$. Now we need to make natural transformations $\eta : G \circ F \Rightarrow \text{Id}$ and $\theta : F \circ G \Rightarrow \text{Id}$. For η we note that we need an arrow $X^{\text{Cof}} \to X$ which we have by the factorization. For θ on the other hand we need for a cofibrant λ-small object X an arrow $X \to X$ which we take to be the identity map. All η_X and $g(\theta_X)$ are weak equivalences, and therefore we can conclude that $\text{hocolim}(\mathscr{M}_\lambda)$ and $\text{hocolim}(\mathscr{M}_\lambda^{Cof})$ are weakly equivalent. □

From this proposition we already get the idea that Proposition 4.2.12 is important. This is because Proposition 4.2.13 allows us to restrict to the generators of the model category, and that already is a step in the right direction.

Now we have sufficient material to prove Proposition 4.2.3. Let λ be a cardinal such that it is big enough to make all the previous propositions hold. Define the following set

$$\mathscr{C}R := \{\gamma^* \in \mathrm{c}\mathscr{M} \mid \gamma^n \in \mathscr{M}_\lambda \text{ for all } n \in \mathbb{N}\}$$

and define $f : \mathscr{C}R \to \mathscr{M}$ which sends γ^* to γ^0. We do not require that γ^* in $\mathscr{C}R$ is cofibrant. Also, let $\mathscr{C}$ be $\mathscr{M}_\lambda^{\text{cof}}$, and note that f lands in $\mathscr{C}$.

Our goal is now to prove that $\operatorname{hocolim}(\mathscr{C}R \times \Delta \downarrow X) \to X$ is a weak equivalence, and for that we consider the following diagram for a fibrant object X

$$\begin{array}{ccccc} \operatorname{hocolim}(\mathscr{C} \downarrow X) & \xleftarrow{f_*} & \operatorname{hocolim}(\mathscr{C}R \downarrow X) & \xrightarrow{i_*} & \operatorname{hocolim}(\mathscr{C}R \times \Delta \downarrow X) \\ & {\scriptstyle a} \searrow & \downarrow & \swarrow {\scriptstyle p} & \\ & & X & & \end{array}$$

If we take λ sufficiently large, then by Proposition 4.2.13 the map a is a weak equivalence. So, to show that p is a weak equivalence, it suffices by the 2-out-of-3 property to show that f_* and i_* are weak equivalences. This is the most technical part of the proof, but the idea is simple. We just apply Proposition 3.3.6, and after sufficient work we get the equivalence.

Lemma 4.2.14. *The map f_* is a weak equivalence.*

PROOF. We need to show that the map $f_* : \operatorname{hocolim}(\mathscr{C}R \downarrow X) \to \operatorname{hocolim}(\mathscr{C} \downarrow X)$ is a weak equivalence, and for that we use Proposition 3.3.6. First, we need to make $\mathscr{R} : \mathscr{C} \downarrow X \to \mathscr{C}R \downarrow X$, and for that look at a certain category. Define $\mathscr{E}$ to be the full subcategory of cosimplicial objects such that each γ^n is in $\mathscr{C}$. Recall that $\mathscr{C}$ consists of the cofibrant replacements of λ-small objects, so every object in $\mathscr{C}$ is cofibrant an λ-small. Our goal is to make a functor R for objects in $\mathscr{E}$ such that

(1) $R(A)$ is Reedy cofibrant;
(2) we have a natural weak equivalence $R(A) \to A$;
(3) $R(A)^0 = \gamma^0$ and $R(A)^0 \to \gamma^0$ is the identity.

This functor is just the Reedy cofibrant replacement functor where we choose the object in zeroth to be γ^0. Now we define two functors where $\mathscr{R}$ maps $C \to X$ to $(R(C), C \to X)$ and $f : \mathscr{C}R \downarrow X \to \mathscr{C} \downarrow X$ maps a pair $(\gamma^*, \gamma^0 \to X)$ to $\gamma^0 \to X$.

Next we need to make a natural transformation $\eta : f \circ \mathscr{R} \Rightarrow \mathrm{Id}$. Note that $F(\mathscr{R}(C \to X)) = f((R(C), C \to X)) = C \to X$, so we can take η to be the identity. Isomorphisms are weak equivalences, so in this case we have a weak equivalence. Also, we make a zig-zag natural transformation $\theta : \mathscr{R} \circ F \Rightarrow \mathrm{Id}$. Define a functor $H((\gamma^*, \gamma^0 \to X)) = (R(\gamma^*), \gamma^0 \to X)$, and we shall make natural transformations $H \Rightarrow \mathscr{R} \circ f$ and $H \Rightarrow \mathrm{Id}$. By the second property of the functor R we have a natural weak equivalence from H to the identity. We have a natural map $\gamma^* \to c^*(\gamma^0)$, because for each n there is a unique $\gamma^n \to \gamma^0$. This gives a map $R(\gamma^*) \to R(c^*(\gamma^0))$, and that way we get a natural transformation $H \Rightarrow \mathscr{R} \circ F$. Applying f to each of these natural transformation gives the identity map at $\gamma^0 \to X$, and therefore all requirements of Proposition 3.3.6 are satisfied. This allows us to conclude that f is a weak equivalence. □

Lemma 4.2.15. *The map i_* is a weak equivalence.*

PROOF. We apply Proposition 4.2.12 which says that it is sufficient to show that $\operatorname{hocolim}(\mathscr{C}R^0 \downarrow X) \to \operatorname{hocolim}(\mathscr{C}R^n \downarrow X)$ is a weak equivalence for all n, and recall that the map $i : \mathscr{C}R^0 \downarrow X \to \mathscr{C}R^n \downarrow X$ maps $(\gamma^*, \gamma^0 \to X)$ to $(\gamma^*, \gamma^n \to \gamma^0 \to X)$. Note that there is a unique map $\gamma^n \to \gamma^0$ in γ^*, so this is well-defined. To show that these two homotopy colimits are weakly equivalent, we apply Proposition 3.3.6.

Next we need to define a functor $j : \mathscr{C}R^n \downarrow X \to \mathscr{C}R^0 \downarrow X$, and for that we make a map $\gamma^0 \to \gamma^n$. Note that we have $d : [0] \to [n]$ in the simplex category Δ which sends 0 to n, and this gives a map $c : \Delta[0] \to \Delta[n]$ of simplicial sets. Also, from d we get a map $\gamma^0 \to \gamma^n$, and thus we can define j as the functor which sends $(\gamma^*, \gamma^0 \to X)$ to

$(\gamma^*, \gamma^n \to \gamma^0 \to X)$. Using the cosimplicial identities we can conclude that $j(i((\gamma^*, \gamma^0 \to X))) = (\gamma^*, \gamma^0 \to X)$, so $j \circ i = \mathrm{Id}$.

The map c gives a map $\Delta[n] \to \Delta[0] \to \Delta[n]$ which we shall call c as well. To make a zig-zag of natural transformations from $i \circ j$ to the identity, we first show that the map c is homotopic to the identity. Now we can consider the following diagram

$$\begin{array}{ccc} \Delta[n] \coprod \Delta[n] & \xrightarrow{(\mathrm{Id}, c)} & \Delta[n] \\ \downarrow & & \downarrow \\ \Delta[n] \times \Delta[1] & \longrightarrow & 1 \end{array}$$

Let ι_k be the inclusion $\Delta[n] \to \Delta[n] \times \{k\}$ for $k \in \{0, 1\}$, and then we can define the arrow $\Delta[n] \coprod \Delta[n] \to \Delta[n] \times \Delta[1]$ as ι_0 on the first copy and ι_1 on the second copy. This is a monomorphism, and thus this map is a cofibration. Also, one can show that $\Delta[n]$ is a Kan complex meaning that the map $\Delta[n] \to 1$ is a Kan fibration. The geometric realization is a special case of $\otimes_\Delta$, namely $|K| = K \otimes_\Delta \Delta^-$, so by Proposition 3.3.4 we can conclude that $|\Delta[n]| = \Delta^n$. Therefore, the map $\Delta[n] \to 1$ induces an isomorphism on all homotopy groups, and therefore $\Delta^n \to 1$ is a trivial fibration. Hence, this diagram has a lift $h : \Delta[n] \times \Delta[1] \to \Delta[n]$ which is a homotopy between c and the identity.

Before continuing we need to introduce a little extra notation. For a cosimplicial resolution γ and a simplicial set K we define $K \otimes \gamma$ to be cosimplicial object which is $(K \times \Delta[n]) \otimes \gamma$ in degree n. Using this notation and the homotopy to define the zig-zag of natural transformations and we define H to be the functor sending $(\gamma^*, \gamma^n \to X)$ to $(\gamma^* \otimes \Delta[1], ((\Delta[1] \times \Delta[n])^0 \otimes \gamma^* \to (\Delta[n])^0) \otimes \gamma^*)$. Using Proposition 3.3.4 we can conclude that

$$((\Delta[1] \times \Delta[n]) \otimes \gamma^*)^0 = (\Delta[1] \times \Delta[n] \times \Delta[0]) \otimes_\Delta \gamma^* = (\Delta[1] \otimes \gamma^*)^n,$$

and that

$$(\Delta[n] \otimes \gamma^*)^0 = (\Delta[n] \times \Delta[0]) \otimes_\Delta \gamma^* = \Delta[n] \otimes \gamma^* = \gamma^n$$

so that H sends elements of $\mathscr{C}R^n \downarrow X$ to $\mathscr{C}R^n \downarrow X$. Recall that we have inclusion maps $\iota_0, \iota_1 : \Delta[1] \to \Delta[n] \times \Delta[1]$, and with these we define two maps from $\Delta^0 \otimes \gamma^* \to \gamma^n$, namely $(h \otimes \iota_0) \otimes \mathrm{Id}$ and $(h \otimes \iota_1) \otimes \mathrm{Id}$. Since $h \circ \iota_0 = \mathrm{Id}$ and $h \circ \iota_1 = c$, we get two natural transformations $\eta : \mathrm{Id} \Rightarrow H$ and $\theta : i \circ j \Rightarrow H$. For η we have the commutative diagram

$$\begin{array}{ccccc} \gamma^n & \xrightarrow{\iota_0 \otimes \mathrm{Id}} & (\Delta^1 \otimes \gamma^*)^n & \xrightarrow{h} & \gamma^n \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

and for θ we have the commutative diagram

$$\begin{array}{ccccc} \gamma^n & \xrightarrow{\iota_1 \otimes \mathrm{Id}} & (\Delta^1 \otimes \gamma^*)^n & \xrightarrow{h} & \gamma^n \\ \downarrow & & & \swarrow & \\ \gamma^0 & \longrightarrow & X & & \end{array}$$

To finish this proof, we need to check that we get weak equivalences if we apply j to the natural transformations. But if we apply j to them, we get the identity at γ^0 which is a weak equivalence. Hence, the map i_* is indeed a weak equivalence. □

Part 2

Model Structures on Toposes

CHAPTER 5

Topos Theory

Topos Theory connects geometry and logic. We have a geometric way to construct toposes, namely by using *sheaves on a site*. Also, from a topos one can construct a model of intuitionistic set theory, and to do so, one requires *elementary toposes*. We are mostly interested in the geometric side of the story, and we will thus restrict ourselves to the less general notion of a *Grothendieck topos*.

5.1. Basic Theory

The story starts with the notion of a *sheaf*. Recall that a presheaf on a category $\mathscr{C}$ is a functor $P : \mathscr{C}^{\mathrm{op}} \to \mathrm{Sets}$. Sheaves arose in geometry as a way to study locality, and the prototypical example of a sheaf is the functor $C(U, \mathbb{R})$ defined on the open subsets U of a topological space. The notion of a sheaf is very diverse, and there are many different ways of thinking about it. One possible way is that it gives a way to solve 'local to global' problems. An example of such a problem, is defining the derivative of a function on a manifold. Since a manifold is locally Euclidian, we can determine the derivative locally using methods from analysis. However, we want to define the derivative on the complete manifold, and thus we need to glue these local solutions together into a global solution.

Let us try to explain this idea in a more general fashion. For every open subset we have a set of 'candidates' at that part, and with these we can do two things. Firstly, we can restrict the possible solutions to smaller subsets, and secondly we can glue them together. To glue the candidates we need them to be consistent in a certain way. This is because the candidates might be defined on overlapping open subsets, and they should not contradict each other. Therefore, we will need that the constructed set of candidates agree on their overlaps. With such a consistent system of candidates, we can glue them together to obtain a solution on their union.

However, this definition only works for topological spaces, because the objects of a general category might not be open subsets. To define the notion of sheaves on an arbitrary category, we will thus need to generalize the notion of a topological space. The main idea behind this generalization is that the required fundamental notion is that of a cover. So, instead of saying which sets are open, we give a collection of open covers of an object.

Let us be more precise and give an actual definition. First, we start with generalizing topologies to *Grothendieck Topologies*. We start with a small category $\mathscr{C}$ and an object C of $\mathscr{C}$, and we would like to say what the open covers of C are. However, before we can do so, we must define the covers of C, and this definition can easily be generalized from the topological example. A cover consists of open sets V_α all contained in some fixed open set, and from this we generalize the notion of a *sieve*. A sieve S on C is a set of arrows into C such that whenever we have arrows $g : Z \to Y$ and $f : Y \to X$ with $f \in S$, then we have $f \circ g \in S$. For an arrow $h : D \to C$ and a sieve S on C we define $h^*(S)$ as the set $\{g : X \to D \mid h \circ g \in J(C)\}$. With this generalized notion of cover we define

Definition 5.1.1 (Grothendieck Topology)**.** Let $\mathscr{C}$ be a small category and let J be a function which assigns to every object C of $\mathscr{C}$ a set $J(C)$ of arrows into C. Then we say J is a *Grothendieck topology* iff the following is satisfied

(1) For every object C the maximal sieve $t_C = \{f : Y \to X\}$ is an element of $J(C)$.
(2) If $S \in J(C)$ and $h : D \to C$, then $h^*(S) \in J(D)$.
(3) Let a sieve $S \in J(C)$ be given, and let R be any sieve on C. If for every $h : D \to C$ in S we have $h^*(R) \in J(D)$, then $R \in J(C)$.

Sieves in $J(C)$ are called *covering sieves*.

The second condition is called stability and the third axiom is called transitivity. A *site* is defined as a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a small category and J is a Grothendieck topology on $\mathscr{C}$. Let us consider some examples of sites.

Example 5.1.2 (Topological Spaces)**.** Given a topological spaces (X, τ), the collection τ is a preorder, and thus a category. First of all, notice that a sieve S is just a downward closed set of opens, because by definition we have that $V \in S$ whenever $U \in S$ and $V \subseteq U$. If we have an arrow $h : V \to U$, then we have

$$h^*(S) = \{W \subseteq V \mid W \in S\} = \{W \cap V \mid W \in S\}$$

where we use that S is downward closed. Given an open U, we define $J(U)$ as $\{S \mid \bigcup S = U\}$. Since the maximal sieve t_U contains U, we must have that $t_U \in J(C)$. If S covers U, then $h^*(S)$ covers V for $h : V \to U$, because we intersect all subsets in S with V. The last property holds as well, because if S covers U locally, then it covers U globally as well.

This example already explains a lot about the definition. The maximal sieve is the biggest cover you can make, and that covers the object. The second property says that if you have a cover for U, then you can restrict that cover to parts V of U by pulling it back to V. The last axiom says that being an open cover is a local property, because whenever you cover U locally, then you cover it globally. The next example gives a Grothendieck topology on something which is not on a topological space.

Example 5.1.3 (Complete Boolean Algebras)**.** Let $\mathbb{B}$ be a complete Boolean algebra which again is a category, because it is a partial order. A sieve S is again a downward closed set of elements, and for $a \leq b$ we have $h^*(S) = \{a \wedge c \mid c \in S\}$ for the same reason as before. Define $J(b) = \{S \mid \bigvee S = b\}$. It is not difficult to verify that this is indeed a Grothendieck topology.

Now we have generalized the notion of topologies so that we can have such things on arbitrary categories. The next step in the story is the definition of a sheaf, and using the more general notion of a Grothendieck topology, we can do this on arbitrary sites.

Definition 5.1.4 (Sheaf)**.** Let $(\mathscr{C}, J)$ be a site, and let $P : \mathscr{C}^{\text{op}} \to \mathbf{Sets}$. For an object C and a covering sieve S we say that a *matching family* consists of elements $x_f \in P(D)$ for $f \in S$ such that for all $f : D \to C$ and $g : E \to D$ with $f \in S$ we have $x_{f \circ g} = P(g)(x_f)$. Then we call P a sheaf iff for every matching family there is a unique $x \in P(C)$ such that for all $f \in S$ we have $x_f = P(f)(x)$. The full subcategory of $\text{Pre}(\mathscr{C})$ consisting of sheaves on the site $(\mathscr{C}, J)$ is denoted as $\text{Sh}(\mathscr{C}, J)$.

Often we will call sheaves F from the French *faisceau*. Because every sheaf is a presheaf, we have the restriction operator, and the sheaf property gives the gluing of candidates. Also, for the same reason, we have a notion of morphisms between sheaves, namely natural transformations, and therefore this gives a category. Let a *Grothendieck topos* be a category which is equivalent to a the category of shaves on a site. An example of a sheaf is the functor $C(U, \mathbb{R})$ on a topological space or $C^\infty(U, \mathbb{R})$ on a smooth manifold. Grothendieck toposes have many useful properties, and one is that they are locally presentable. To prove this, one can use Theorem 2.2.8 and Giraud's theorem [**MLM92**].

Note that both $C(U, \mathbb{R})$ and $C^\infty(U, \mathbb{R})$ are not just a sets, but actually they are abelian groups. Such things happen in many algebraic examples where the sheaves are structured like abelian groups or simplicial sets. To give a general definition of a structured sheaf actually requires rewriting the definition, but for certain structures (which are defined using finite limits) this is not necessary. Since all our structured sheaves will be such simple sheaves, we will thus go for a more specific definition. An *abelian sheaf* is an abelian group object of sheaves, and a *simplicial sheaf* is a simplicial object in the category of sheaves.

It is not difficult to see that the category of presheaves is complete and cocomplete, because we can take limits and colimits of presheaves pointwise. Actually, the category of presheaves has more nice structure, namely it has a *subobject classifier* which are defined as

Definition 5.1.5 (Subobject Classifier)**.** Let $\mathscr{C}$ be a category with all finite limits, and let Ω be an object. Then we say Ω is a *subobject classifier* iff we have an arrow $t : 1 \to \Omega$ for every monomorphism $A \hookrightarrow B$ there is a unique arrow $\chi_A : B \to \Omega$ turning the following diagram into a pullback

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle t} \\ B & \underset{\chi_A}{\longrightarrow} & \Omega \end{array}$$

So, subobjects of X correspond with arrows from X to Ω. A subobject classifier can be interpreted as an internal notion of truth, and the arrow t gives the true global 'element' of Ω. Also, if we have a subobject classifier Ω in a cartesian closed category, then we can define the power $P(X)$ of an object X to be Ω^X.

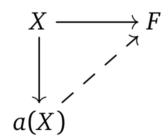
In **Sets** the subobject classifier is $\{0, 1\}$ and the map from 1 to $\{0, 1\}$ sends the point to 1. A predicate on some set X can be identified as a subset of it, namely the subset of all objects in X for which the property holds. We see the inclusion $A \subseteq B$ as a predicate on B, and the pullback says that we have an arrow $\phi : B \to \Omega$ such that $A = \{x \in B \mid \phi(x) = 1\}$. Replacing ϕ by the formula $\phi(x) = 1$, then A is precisely the subset of B defined by this formula. In this case the power of an object X is just the power set $\mathscr{P}(X)$.

The subobject classifier of $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ is defined to be the collection of all sieves on an object. Lastly, the category of presheaves is cartesian closed where for presheaves X and Y we define

$$Y^X(C) = \mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}(X \times y_C, Y).$$

However, this only gives structure on the category of presheaves, and we would like structure on the category of sheaves. It turns out that the product of sheaves, exponential of sheaves and that the subobject classifier Ω are sheaves. However, for the colimits this does not hold, and for that we need more namely *sheafification functor* a. Sheafification will not be a main topic in this thesis, and its definition is rather technical, so we will skip the details here. The important property of sheafification a is

Proposition 5.1.6. *We have a map $X \to a(X)$ and that for a presheaf X. Also, for every sheaf F, maps from X to F factor uniquely through the map $X \to a(X)$. In a diagram this is*



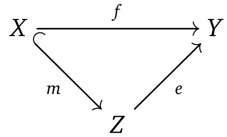
Now we define colimits in the category of sheaves on a site by taking the colimit as presheaves and then applying sheafification. Sheafification is an important functor in sheaf theory, because it preserves finite limits and it is the left adjoint of the inclusion functor from $\mathrm{Sh}(\mathscr{C}, J) \to \mathrm{Pre}(\mathscr{C})$. All Grothendieck toposes can thus be written as a reflective subcategory of a presheaf category where left adjoint preserves finite limits. Giraud's theorem gives a converse of this statement, and says that every reflective subcategory of $\mathrm{Pre}(\mathscr{C})$ is the category of sheaves on some site.

The following step is to define a suitable notion of morphism between Grothendieck toposes, and the main idea is to imitate the case of toplogical spaces. Suppose we have two topological spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$. Given a continuous map f from X to Y, then we can construct a map from $\mathrm{Sh}(\mathscr{O}_X)$ to $\mathrm{Sh}(\mathscr{O}_Y)$ with their standard Grothendieck topologies. If we have an open U in Y and a sheaf F on $\mathscr{O}_X$, then we define $f_*(F)(U) = F(f^{-1}(U))$. This is called the *direct image* of the sheaf. Using a bit more work and technique, one can show that f_* has a left adjoint f^* which preserves finite limits. We generalize this construction to obtain a suitable notion of morphisms between Grothendieck toposes, namely the notion *geometric morphism*.

Definition 5.1.7 (Geometric Morphism)**.** Let $\mathscr{E}$ and $\mathscr{F}$ be Grothendieck toposes. A *geometric morphism* from $\mathscr{E}$ to $\mathscr{F}$ is an adjunction $f^* \dashv f_*$ where $f^* : \mathscr{F} \to \mathscr{E}$, $f_* : \mathscr{E} \to \mathscr{F}$ and f^* preserves finite limits. We call f_* the *direct image* part and f^* the *inverse image* part.

The direction of the morphism is f_*. A geometric morphism is called *surjective* if f^* is faithful. The last important property is that we can factorize every map between sheaves as an epimorphism followed by a monomorphism. This means that we can talk about the image of a map.

Proposition 5.1.8. *Let $\mathscr{E}$ be a topos and let $f : X \to Y$ be a map. Then we can factorize f as follows*



where m is mono and e is epi. Also, m is the image of f which means that whenever f factorizes through some mono h, then m factorizes through h as well.

PROOF. The first step is to construct the cokernel pair of f, and for that we look at the following pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ Y & \xrightarrow[p]{} & Z \end{array}$$

The corkernel pair of f is p and q. Next we take the equalizer of p and q, and we get a monomorphism $m : E \to Y$. By the universal property of the equalizer, we get an arrow $e : X \to E$. This gives the factorization of f as $m \circ e$.

Now we first show that whenever we can write $f = h \circ g$ with h mono, we can write $m = k \circ h$. We now have the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{e} & E & \xrightarrow{m} & Y & \overset{q}{\underset{p}{\rightrightarrows}} & Z \\ \Big\| & & & & \Big\| & & \\ X & \xrightarrow{g} & F & \xrightarrow{h} & Y & & \end{array}$$

First, we show that h is the equalizer of two arrows, namely $u = \chi_F$ and $v = t \circ !_Y$ where $!_Y$ is the unique map from Y to 1. By the universal property of Ω we have the following pullback square

$$\begin{array}{ccc} F & \xrightarrow{!_F} & 1 \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ Y & \xrightarrow[\chi_F]{} & \Omega \end{array}$$

For an arbitrary object W we have that an arrow φ from W to F corresponds to an arrow $\psi : W \to Y$ such that $\chi_F \circ \psi = t \circ !_W$. Now note that $!_W = !_Y \circ f$, so for ψ we have that $\chi_F \circ \psi = t \circ !_Y \circ \psi$. Hence, from this we can conclude that h is the equalizer of $t \circ !_Y$ and χ_F.

Because $f = h \circ g$ and $u \circ h = v \circ h$, we also have $u \circ f = u \circ h \circ g = v \circ h \circ g = v \circ f$. Now we look at the following diagram

$$\begin{array}{cccc} X & \xrightarrow{f} & Y & \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle q} & \searrow{\scriptstyle v} \\ Y & \xrightarrow{p} & Z & \\ & \searrow_{u} & & \Omega \end{array}$$

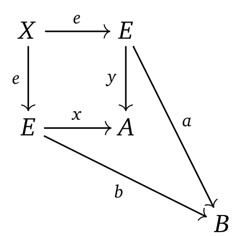
By the universal property of the pushout we thus get an arrow $w : Z \to \Omega$ such that $w \circ p = u$ and $w \circ q = v$. From that we can conclude that $m \circ u = m \circ w \circ p = m \circ w \circ q = m \circ v$, and now we consider the diagram

$$\begin{array}{ccccc} F & \xrightarrow{h} & Y & \overset{u}{\underset{v}{\rightrightarrows}} & \Omega \\ & {\scriptstyle m}\nearrow & & & \\ E & & & & \end{array}$$

By the universal property of the equalizer we then get an arrow $k : E \to F$ which gives the desired factorization. Hence, we have $m = k \circ h$, so m factors through all monos h for which we can write $f = h \circ g$.

To conclude the argument, we need to show that e is an epimorphism. For this we factorize $e = m' \circ e'$ in the same way as before with m' mono. So, f is now equal to $m \circ m' \circ e'$, and because m factors through monos h with $f = h \circ g$, we can write $m = m \circ m' \circ k$ for some k. Now we have that $m \circ \mathrm{Id} = m \circ m' \circ k$, and because m is mono, this means that $\mathrm{Id} = m' \circ k$. The arrow m' is a retract, so it must be epi. Also, we have $m' \circ (k \circ m') = m' \circ \mathrm{Id}$, and because m' is mono, this gives that $k \circ m' = \mathrm{Id}$. Hence, m' is an isomorphism, and from this we will conclude that e is epi.

For the construction of m' we first took the cokernel pair of e, which was $x, y : E \to A$, and then we defined m' as the equalizer of x and y. Since m' is iso, we get that $x = y$. To show that e is epi, we take arrows $a, b : E \to B$ such that $a \circ e = b \circ e$.

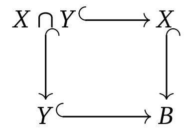


By the universal property of the pullback we get a unique arrow u such that $a = u \circ x$ and $b = u \circ y$. Using that $x = y$ we get $a = u \circ x = u \circ y = b$, so $a = b$. This gives that e is epi, and now we have the required factorization. □

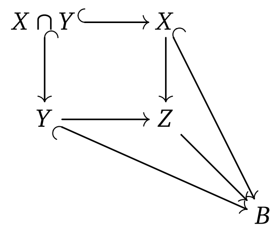
5.2. Logic in Toposes

The subobject classifier allows us to do logic in toposes. Normally in logic notions of truth have a certain structure, namely we can take conjunctions, disjunctions and so on, and our goal is to show that we have such structure on Ω as well. More precisely, we want to construct morphism $\wedge, \vee, \to : \Omega \times \Omega \to \Omega$ and $\bot, \top : 1 \to \Omega$ which represent the logical connectives. However, if we just look at the topos itself, it is not obvious what kind of properties these maps should satisfy or how we should define them. The point is that we should look from two perspectives to the topos, namely we should look both from internal and external perspective. Objects and maps are internal notions, but Hom-sets and sets of subobjects are external notions, because these do not live in the topos itself. To construct the maps, we first study the topos from external perspective where we can talk about the subobjects of some objects. There we can easily find the required structure, and we have an obvious way of understanding why they mean the right thing. Next we make this structure internal which can be done by the universal property of Ω.

Let us start by studying the external notion. Recall that $\mathrm{Sub}(B)$ consists of isomorphism classes of monomorphisms $X \to B$, and it can be ordered by saying that $i \leq j$ iff we can factor i through j. Suppose that we have two subobjects $X \hookrightarrow B$ and $Y \hookrightarrow B$. We can find a greatest lower bound of them by taking the pullback of the square



One can directly show by definition that the map $X \cap Y \to B$ is a monomorphism, and thus $X \cap Y$ is a subobject of B. A greatest upper bound can be found by taking the pushout of the square



The map $Z \to B$ is a monomorphism which is proved in [**Joh14**]. However, this result requires more technique, and even without it we can define the union. Using Proposition 5.1.8 we can write f as $Z \longrightarrow X \cup Y \hookrightarrow B$ where $X \cup Y$ is indeed a subobject of B. From the construction it follows that $X \cap Y$ and $X \cup Y$ are indeed the greatest lower bound and the least upper bound respectively in the partially ordered set of subobjects. We can thus conclude that $\mathrm{Sub}(B)$ always is a lattice.

When B is the terminal object 1, there is more structure on $\mathrm{Sub}(1)$, namely it has exponentials. If we have two subobjects $i : X \to 1$ and $j : Y \to 1$, we claim that X^Y is a subobject of 1 as well. We have a unique map k from X^Y to 1, and now we show that k is mono. Suppose that we have $f, g : Z \to X^Y$ such that $k \circ f = k \circ g$. Using the adjunction of the exponential f and g correspond to maps $\widetilde{f}, \widetilde{g} : X \times Y \to X$ such that $i \circ \widetilde{f} = i \circ \widetilde{g}$. Since i is mono, we get that $\widetilde{f} = \widetilde{g}$, and thus $f = g$. By the universal property of the exponential, we can thus conclude that $\mathrm{Sub}(1)$ also has an implication. Lastly, we have $0 \to 1$ and an arrow $1 \to 1$ which represent the bottom and the top element of the partial order respectively.

Now we know that $\mathrm{Sub}(1)$ has the desired structure, but this is constructed from an external perspective. This is not precisely what we want, because we would like to construct the structure internally using arrows in the topos. By definition of Ω we have that $\mathrm{Sub}(X)$ and $\mathrm{Hom}(X, \Omega)$ are naturally isomorphic. Using this isomorphism we can transfer the structure from $\mathrm{Sub}(X)$ to $\mathrm{Hom}(X, \Omega)$ which gives us the respective operations on $\mathrm{Hom}(X, \Omega)$. Now we will internalize the operations using the Yoneda lemma. All of them can be done in a similar way, and we do it for the conjunction.

Since we have natural isomorphisms $\mathrm{Sub}(X) \times \mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega) \times \mathrm{Hom}(X, \Omega)$ and $\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$, and a natural transformation $\mathrm{Sub}(X) \times \mathrm{Sub}(X) \to \mathrm{Sub}(X)$, we get a natural transformation $\wedge : \mathrm{Hom}(X, \Omega) \times \mathrm{Hom}(X, \Omega) \to \mathrm{Hom}(X, \Omega)$. The map $\mathrm{Sub}(X) \times \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ mapping (X, Y) to $X \cap Y$ is natural by the universal property of the pullback, and for $X \cup Y$ and X^Y similar arguments can be given. Since $\mathrm{Hom}(X, \Omega) \times \mathrm{Hom}(X, \Omega)$ is naturally isomorphic to $\mathrm{Hom}(X, \Omega \times \Omega)$, we have a natural map $\mathrm{Hom}(X, \Omega \times \Omega) \to \mathrm{Hom}(X, \Omega)$, and this gives a map from $y_{\Omega\times\Omega} \to y_\Omega$. Applying the Yoneda lemma gives a map in $y_\Omega(\Omega \times \Omega) = \mathrm{Hom}(\Omega \times \Omega, \Omega)$ which is called the internal meet operator. To summarize, we have proven the following theorem

Theorem 5.2.1. *The subobject classifier Ω in every topos is an internal Heyting algebra.*

In logic we also need quantifiers to get an expressive language. However, to define these an intermediate step is required, namely we need to see quantifiers as adjoints. Using this view, we can define the quantifiers on toposes using the same techniques as for Theorem 5.2.1.

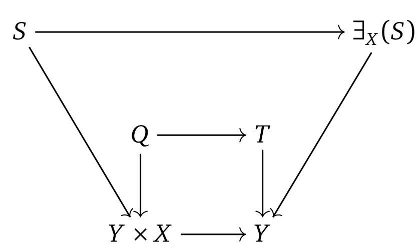
Let us start by explaining in **Sets** how we can see quantifiers as adjoint. Using the quantifiers $\exists_X$ and $\forall_X$ we can turn a formula with free variables X and Y to a formula whose only free variable is Y, so they can be seen as maps $\exists_X, \forall_X : \mathscr{P}(X \times Y) \to \mathscr{P}(Y)$. More explicitly, they are defined as $\exists_X(S) = \{y \mid \text{there is } x \text{ with } \langle x, y\rangle \in S\}$ and $\forall_X(S) = \{y \mid \text{for all } x \text{ we have } \langle x, y\rangle \in S\}$ On the other hand, we have a function $p^* : \mathscr{P}(Y) \to \mathscr{P}(X \times Y)$ sending a set T to $\{\langle x, y\rangle \mid y \in T\}$. It is not difficult to show that for all subsets $S \subseteq X \times Y$ and $T \subseteq Y$ we have $\exists_X(S) \subseteq T$ iff $S \subseteq P^*(T)$ and that $T \subseteq \forall_X(S)$ iff $p^*(T) \subseteq S$. This means that the existential quantifier is the left adjoint of the map $\mathscr{P}(p^*)$ and the universal quantifier is the right adjoint of it.

In the motivation we defined the quantifiers for all powersets, and for toposes we can define them as well for all 2^X. However, we are only interested in quantifiers on the subobject classifier, so we will only define it for Ω^1.

Theorem 5.2.2. *Let X be an object in a topos. Then the map $P(!_X)$ has a left adjoint $\exists_X$ and a right adjoint $\forall_X$.*

PROOF. The proof uses similar techniques as Theorem 5.2.1, and we will only do it for the existential quantifier. For the right adjoint $\forall_X$ we can give a similar argument, but then we need that the functor $\mathscr{E}/1 \to \mathscr{E}/X$ has a right adjoint and that we have maps $\mathrm{Sub}(X) \to \mathscr{E}/X$ and $\mathrm{Sub}(1) \to \mathscr{E}/1$. This construction of the right adjoint is more complicated, and for that we refer the reader to [**MLM92**]. Our goal is to construct a map $P(X) \to P(1)$, and to do this in the same way as before, we need to construct maps $\mathrm{Hom}(Y, P(X)) \to \mathrm{Hom}(Y, \Omega)$ for every object Y. Since $\mathrm{Hom}(Y, P(X)) \cong \mathrm{Hom}(Y \times X, \Omega)$, we have $\mathrm{Hom}(Y, P(X)) \cong \mathrm{Sub}(Y \times X)$. So, if we can make a natural left adjoint $\mathrm{Sub}(Y \times X) \to \mathrm{Sub}(Y \times 1)$, then we can apply Yoneda to conclude the argument.

Now we can construct $\exists_X$ externally. Note that externally we have a map $\mathrm{Sub}(Y) \to \mathrm{Sub}(X \times Y)$ given by pullback, and we need to prove that this map has a left adjoint. For a subobject $S \to Y \times X$ we get a map $S \to Y$ which we can factor as $S \to \exists_X(S) \to Y$ as in Proposition 5.1.8. We can form the following diagram where Q is the pullback



Note that this construction is natural, and that is because $\exists_X(S)$ is universal by Proposition 5.1.8. If we have $f : Z \to X$, we get $f^*(S)$ by pullback. Then we have two factorizations of $f^*(S) \to Y$, namely $f^*(S) \to \exists_Z(f^*(S)) \to Y$ and $f^*(S) \to S \to \exists_X(S) \to Y$, and the arrows $\exists_X(S) \to Y$ and $\exists_Z(f^*(S)) \to Y$ both are mono. Hence, we get a map $\exists_Z(f^*(S)) \to \exists_X(S)$ by Proposition 5.1.8.

If we have a map $\exists_X(S) \to T$, then we get by the universal property of the pullback a unique map $S \to Q$. On the other hand, if we have a map $S \to Q$, then we have a factorization of the map $S \to Y$ as $S \to T \to Y$, and the last arrow here is a monomorphism. Hence, by Proposition 5.1.8 we thus get a unique map $\exists_X(S) \to T$. □

5.3. A Short Intermezzo on Localic Toposes

Our next goal is to prove Barr's theorem which says that for every topos $\mathscr{E}$ there is a surjective geometric morphism $\mathrm{Sh}(\mathscr{B}) \to \mathscr{E}$ where $\mathbb{B}$ is a complete Boolean algebra. The proofs of the results require some methods which are not needed in the remainder of this thesis, and thus we will not prove them. To prove this, we need to make use of *locales*.

Definition 5.3.1 (Locale)**.** A *locale* is a lattice with all joins and finite meets such that $a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i)$.

With locales we want to imitate topological spaces, so we need a functor from **Top** to locales which sends $(X, \mathscr{O}_X)$ to $\mathscr{O}(X)$ and $f : X \to Y$ to $f^{-1} : \mathscr{O}(Y) \to \mathscr{O}(X)$. Hence, we define a morphism from a locale X and to a locale Y to be a map $f : Y \to X$ such that

$$f(0) = 0, \quad f(1) = 1, \quad f(a \wedge b) = f(a) \wedge f(b), \quad f(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} f(a_i).$$

Now we indeed have this functor. The next step is to define a Grothendieck topology on locales, and we say that a sieve S on c is covering iff its supremum is equal to b. This really copies the Grothendieck topology of a topological spaces, and using the same arguments we can show that it is indeed so.

Theorem 5.3.2. *The following two statements are equivalent*

(1) *There is a site for $\mathscr{E}$ which is a locale with its canonical topology.*
(2) *The topos $\mathscr{E}$ has a site which is a partial order.*

If we want to make a topos with a locale as a site, then it suffices to make one where the site is a partial order. Working with locales is easier than working with general sites, and thus this theorem already simplifies a lot. The next relevant proposition says that construct maps between sheaves, it suffices to make maps between locales.

Proposition 5.3.3. *A map $f : X \to Y$ between locales gives a geometric morphism* $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

Arrows between locales are reversed, so notions of product and coproduct are reversed. For example, to construct the coproduct of locales, we take their product, and then the operations are defined pointwise. Similarly, the notion of epimorphism and monomorphism are reversed. So, if we want to construct an epimorphism from a locale Y to X, then we need to make an injective map from X to Y.

Proposition 5.3.4. *Let X be a locale. Then there is a surjection $Y \to X$ where Y is a complete Boolean algebra.*

PROOF. Note that in a locale we have 0 and 1, and that we can define $U \Rightarrow V$ as $\bigvee_{W \wedge U \leq V} W$. As usual, if we have an implication and a 0, then we can define $\neg U = U \Rightarrow 0$. Now let U be any element of X, and consider

$$X - U = \{V \in X \mid V \geq U\}$$

consisting of the elements greater than U. This is a locale as well where we take the same operations as in X, the 0-element is U and the 1-element is X. Hence, in this locale we have a implication operator $\Rightarrow$ which is defined as $V \Rightarrow U$.

For the construction of Y, we need something which turns a locale into a Boolean algebra. Let us now look at the fix points of $\neg\neg$, so we define for any locale Z

$$Z_{\neg\neg} = \{V \in Z \mid V = \neg\neg V\}.$$

Note that $\neg\neg 0 = 0$ and $\neg\neg 1 = 1$. Also, we can show that if $\neg\neg U = U$ and $\neg\neg V = V$, then $\neg\neg(U \wedge V) = U \wedge V$. For this we need to show laws like $\neg\neg(U \wedge V) = \neg\neg U \vee \neg\neg V$ which require some calculations. By the defining property $Z_{\neg\neg}$ is a complete Boolean algebra, because $\neg\neg U = U$.

Now we can define Y as $\prod_{U \in X}(X - U)_{\neg\neg}$, and note that this is a complete Boolean algebra, because it is the product of complete Boolean algebras. It remains to make an injective map $p : X \to Y$, and for this we need to make maps $p_U : X \to (X - U)_{\neg\neg}$. Since $\neg\neg\neg\neg V = \neg\neg V$ and $V \vee U \geq U$, we define $p_U(V) = \neg\neg(V \vee U)$. To show that p is injective, we take $V \neq W$. Because p is a homomorphism, we have $p(V \vee W) = p(V) \vee p(W)$, so if we show that $p(V \vee W) \neq p(V)$, then we must have $p(V) \neq p(W)$. Since $V \leq V \vee W$ and $V \neq V \vee W$, we have $p_V(V) = 0$ and $p_V(V \vee W) \neq 0$. This is because $V \vee W \leq \neg\neg(V \vee W) = p_V(V \vee W)$ taking the negation in $(X - V)$. Concluding, we have a complete Boolean algebra Y and a surjective map $Y \to X$ of locales. □

5.4. Boolean Localization

Theorem 5.4.1 (Barr's Theorem). *If $\mathcal{E}$ is a Grothendieck topos, then there is a complete Boolean algebra $\mathbb{B}$ and a surjective geometric morphism* $\mathrm{Sh}(\mathbb{B}) \to \mathcal{E}$.

To prove this, we use the theory of localic toposes as discussed in the previous section and the following lemma

Lemma 5.4.2. *Given a Grothendieck topos $\mathcal{E}$, we have a surjective geometric morphism* $\mathrm{Sh}(X) \to \mathcal{E}$ *where X is a locale.*

From this proposition Barr's theorem follows. For a Grothendieck topos $\mathcal{E}$ we have $\mathrm{Sh}(\widetilde{X}) \to \mathrm{Sh}(X) \to \mathcal{E}$ where $\widetilde{X}$ is the completion of X as in Proposition 5.3.4. We have the map $\mathrm{Sh}(\widetilde{X}) \to \mathrm{Sh}(X)$ by Proposition 5.3.3.

Proof of Lemma 5.4.2. Using Theorem 5.3.2 it is sufficient to construct a topos on a site which is a partial order. We just show how to construct the site, and refer the reader to [**MLM92**] for the functor. Let $\mathbb{C}$ be a site with a Grothendieck topology J on it. We define a partial order String($\mathbb{C}$) where the objects are sequences of $C_n \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_0} C_0$, and the order is the prefix. So, we say $t \leq s$ iff s is of the form $C_n \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_0} C_0$, and t is of the form $C_{n+m} \longrightarrow \dots \longrightarrow C_n \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_0} C_0$. This gives a category String($\mathbb{C}$), and we define a functor $\pi : \mathrm{String}(\mathbb{C}) \to \mathbb{C}$ which sends the string $C_n \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_0} C_0$ to C_n. Next we define a Grothendieck topology K on it, so let U be a sieve on s. Then we say that U is covering iff for all $t \leq s$ the set $\{\pi(t' \leq t) \mid t' \in U\}$ covers $\pi(t)$.

Let us prove that K is indeed a Grothendieck topology. The maximal sieve on s is the set $\{t' \mid t \leq s\}$, so it contains the identity for all t. Hence, it is covering by definition. Next let us check stability. Let U be a covering sieve on s, and let $t \leq s$. We need to check that $t^*(U) = \{t' \leq t \mid t' \in U\}$ is a covering sieve. But this follows readily from the definition, because we need to check that for $t' \leq t$ the set $\{\pi(t'' \leq t') \mid t'' \in U\}$ covers $\pi(t')$. This follows from the assumption that U covers s.

Lastly, we need to check the transitivity. Let a covering sieve U on s be given and let V be any sieve on s. Suppose that for all $t \in U$ the sieve $t^*(V)$ is covering. To show that V is covering, we need to show that for all $t \leq s$ the sieve $\{\pi(t' \leq t) \mid t' \in V\}$ covers $\pi(t)$. For every $t' \leq t$ with $t' \in U$ we know that $(t')^*(\{\pi(t' \leq t) \mid t' \in V\})$ is covering, because $(t')^*(V)$ is. Hence, by applying transitivity of J we get that V is indeed covering. □

CHAPTER 6

Some Categorical Logic

6.1. Interpreting Logic in Toposes

An important part of logic is model theory. In the beginning of logic we start by defining a formal language, and we study formal systems. However, if we just study the formal systems by themselves, we do not get the complete story. To prove the undecidability of some formula, we need to consider models and interpretations of formal systems. Normally these interpretations are in **Sets**, but this can be generalized. Instead of considering just set-based interpretations, we can try to interpret them in more general structured categories. At first this seems to be a generalization just for the beauty of generalization, but it is more. For example, Cohen's forcing argument from [**Coh63**] can be formulated in the language of topos theory, and the topos theoretic proof helps revealing the mathematical ideas of it. Therefore, it is nice to be able to interpret logic in arbitrary categories instead of just the category of sets.

Before diving into formal definitions, let us think about the main ideas. The language of categories is formulated using the arrows, and the statements we can formulate are that certain arrows are equal. It does not make sense to say that certain objects are equal: we just talk about arrows and their equality. Hence, to interpret statements in categories, we will need to formulate everything using arrows. The languages we consider consist of multiple types, functions, relations and constants. Normally we see types as sets, and these will be replaced by objects. Interpreting functions is easy, because we can see a function as an arrow. However, for functions with multiple arguments, we will need that the category has products, because then the domain is the product of the types of every argument. For relations we need to think a little. Every predicate on a set X can be identified with a subset of X, namely as all elements for which the property holds, and this can be generalized. Namely, we can interpret predicates on an object X as subobjects of it. If we want to form more complicated statements, namely conjunctions or disjunctions of predicates, we will need that our category has more structure. This is the main starting idea of categorical logic: we interpret statements of some formal systems in an arbitrary category using the arrows.

However, there is a minor problem. In logic we would like to talk about the truth of statements. A formal system does not only consist of a language, but also of axioms which every model of it should satisfy. To do so, we will need an internal logic of truth, and this is given by a subobject classifier defined in definition 5.1.5. Therefore, toposes will allow the desired interpretations.

Now we have enough to study categorical logic, and we start by recapitulating some well-known definitions from logic

Definition 6.1.1 (Language for First-order Many-Sorted Logic)**.** A *language* for first-order many-sorted logic consists of a set $\mathscr{T}$ of types, a set $\mathscr{R}$ of relation symbols, and a set $\mathscr{F}$ of function symbols. Also, for every relation $R \in \mathscr{R}$ we have an arity $\#(R) \in \mathbb{N}_{\geq 1}$

and a type $t(R) \in \mathcal{T}^{\#(R)}$. For every function $f \in \mathcal{F}$ we also have an arity $\#(f) \in \mathbb{N}$ and a input type $i(f) \in \mathcal{T}^{\#(r)}$ and an output type $o(f) \in \mathcal{T}$.

Instead of the notation we write that $f : T_1 \times \ldots T_{\#(f)} \to T$ and $R \subseteq T_1 \times \ldots T_{\#R}$. As always we can consider constants the be functions with arity 0. From the functions we can construct terms by composing them and inserting free variables. Every such term has a type, and the basic formulas are made by putting such terms in relations. From such basic formulas, we can use connectives like $\wedge, \vee, \forall, \exists$ to make more complicated formulas called *sentences*. A more precise definition of formulas can be found in [**Mar02**]. Lastly, a collection of sentences in a certain language gives an *axiom system*. A *structure* S consists of a language L with an axiom system in L.

Example 6.1.2 (Simplicial Sets)**.** Simplicial set can be described using this language. For every natural number i we have a type T_i, and for every arrow $[i] \to [j]$ in Δ^{op} we have an function symbol $f : T_i \to T_j$. We have special arrows namely the boundary maps d_i and the degeneracy maps s_i, and for it to be a simplicial object certain identities using the maps d_i and s_i need to be satisfied. These identities are described in [**GJ09**]. Now we can interpret this in **Sets** in the usual fashion: every type T_i gets interpreted a set X_i and every function symbol $f_i : T_i \to T_j$ gets interpreted as an actual function $f_i : X_i \to X_j$. The axioms say that it is indeed a simplicial set.

As described the next step is to interpret such axiom systems in more general categories than just **Sets**. Types T_i in $\mathcal{T}$ are interpreted as objects $[\![T_i]\!]$. A function $f : T_1 \times \ldots \times T_{\#(f)} \to T$ is interpreted as an arrow $[\![T_1]\!] \times \ldots \times [\![T_{\#(f)}]\!] \to [\![T]\!]$, and a relation $R \subseteq T_1 \times \ldots \times T_{\#(f)}$ is interpreted as a subobject $[\![R]\!] \rightarrowtail [\![T_1]\!] \times \ldots \times [\![T_{\#(f)}]\!]$. Now we can interpret simplicial objects in every category using these definitions, but we cannot say yet whether the axioms are satisfied. In the general case this is difficult, but in this case it is easy. If we write out the axioms of simplicial sets, then we see that only some diagrams need to commute, and that can be translated to the interpretation.

Now let $\mathcal{E}$ be a topos. If we have a relation $R \subseteq T_1 \times \ldots \times T_{\#(f)}$, we can interpret it as a subobject of $[\![R]\!] \rightarrowtail [\![T_1]\!] \times \ldots \times [\![T_{\#(f)}]\!]$ and this gives an arrow $[\![T_1]\!] \times \ldots \times [\![T_{\#(f)}]\!] \to \Omega$. By Theorems 5.2.1 and 5.2.2 we can now talk about formulas $\mathcal{E}$. For example, if we have two formulas $\varphi, \psi : X \to \Omega$, then we can take their conjunction $X \xrightarrow{\varphi \times \psi} \Omega \times \Omega \xrightarrow{\wedge} \Omega$ and the same can be done with the other connectives. A formula said to be *true* iff it factors true the map $t : 1 \to \Omega$. The map t can be seen as an 'element' of Ω, which is the truth element, and that is the motivation of this definition. Now we can define the interpretation of a structure in a topos. If S is a structure, then an interpretation of S in $\mathcal{E}$ is an interpretation on $\mathcal{E}$ such that all axioms are true. From this we can make a category $\text{Mod}_S(\mathcal{E})$. We define a map from $[\![\cdot]\!]_M$ to $[\![\cdot]\!]_N$ to be a collection of arrows $\varphi_i : [\![T_i]\!]_M \to [\![T_i]\!]_N$ such that for each function symbol $f_i : T_1 \times \ldots T_n \to T_0$ the diagram

$$\begin{array}{ccc} [\![T_1]\!]_M \times \ldots \times [\![T_n]\!]_M & \xrightarrow{[\![f_i]\!]_M} & [\![T_0]\!]_M \\ \Big\downarrow{\scriptstyle \varphi_1 \times \ldots \varphi_n} & & \Big\downarrow{\scriptstyle \varphi_0} \\ [\![T_1]\!]_N \times \ldots \times [\![T_n]\!]_N & \xrightarrow[{[\![f_i]\!]_N}]{} & [\![T_0]\!]_N \end{array}$$

commutes, and for each relation symbol R_i of type $T_1 \times \ldots \times T_n$ the object $(\varphi_1 \times \ldots \times \varphi_n)([\![R_i]\!]_M)$ is a subobject of $[\![R_i]\!]_N$. The first requirement just says that it preserves function symbols, and the second requirement says that it preserves relation symbols. Similarly, if $\mathscr{A}$ is a collection of sentences in the language of S, then we define $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E})$ to be the full subcategory of $\mathrm{Mod}_S(\mathscr{E})$ consisting of the objects which satisfy all axioms in $\mathscr{A}$.

For Chapter 7 we will need the notion of a structure defined using finite limits, and for this we need some extra requirements on the structure.

Definition 6.1.3 (Cartesian Logic)**.** Let S be a structure. Then we say S is *defined in terms of finite limits* or that S is defined in *cartesian logic* iff there are no relation symbols except for equality, and all axioms are of the form $\forall_{x_1} \ldots \forall_{x_n}[\varphi \Rightarrow \exists!_{y_1} \ldots \exists!_{y_m} \psi]$ where φ and ψ are formulas only using $\wedge$.

Here $\exists!_x \varphi$ means that there is a unique x which satisfies φ. In the same way as in Example 6.1.2 we can define simplicial objects in an topos to be the interpretations of this structure. One can check that this structure is defined in finite limits by looking at the axioms. Another example of structures defined in finite limits are universal algebras. A universal algebra consists of a collection of function symbols, some of which might have arity 0. A more precise definition can be found in [**SB**]. Note that the interpretation of a function symbol of arity 0 is just an element of a set. The algebra might have some equations which should be satisfied, and all of these can be written in cartesian logic. Hence, every universal algebra is a structure which is definable using finite limits.

From [**Bek01**] we several properties of the category $\mathrm{Mod}_S(\mathscr{E})$.

Proposition 6.1.4. *Let S be a structure defined using finite limits, and let $\mathscr{E}$ be a topos. Then*

(1) $\mathrm{Mod}_S(\mathscr{E})$ *is locally presentable.*
(2) *A geometric morphism $\mathscr{E} \to \mathscr{F}$ induces an adjunction between* $\mathrm{Mod}_S(\mathscr{E})$ *and* $\mathrm{Mod}_S(\mathscr{F})$.
(3) *For a small category $\mathscr{D}$ we have that* $\mathrm{Mod}_S(\mathscr{E}^D)$ *is isomorphic to* $(\mathrm{Mod}_S(\mathscr{E}))^D$.
(4) *There exists a finite limit structure* $\mathrm{Mor}(S)$ *and a canonical equivalence between* $\mathrm{Mor}(\mathrm{Mod}_S(\mathscr{E}))$ *and* $\mathrm{Mod}_{\mathrm{Mor}(S)}(\mathscr{E})$.

PROOF. Most of the proofs are rather easy and straightforward. For example, (2) follows by restricting the adjunction and noting that the left adjoint preserves finite limits. To show (3) one needs to notice that limits and colimits in $\mathrm{Mod}_S(\mathscr{E})^{\mathscr{D}}$ are taken pointwise, and this gives the isomorphism between $\mathrm{Mod}_S(\mathscr{E})^{\mathscr{D}}$ and $\mathrm{Mod}_S(\mathscr{E}^{\mathscr{D}})$.

However, (1) is more difficult, and requires some technique. Our approach will be as follows: first we show it for sets. Now we can conclude using (3) and Example 2.2.9 that it also holds for arbitrary presheaf toposes. If we have a sheaf topos $\mathrm{Sh}(\mathscr{C})$, then we have an adjunction $\mathrm{Sh}(\mathscr{C}) \longleftrightarrow \mathrm{Pre}(\mathscr{C})$. This induces an adjunction $\mathrm{Mod}_S(\mathrm{Sh}(\mathscr{C})) \longleftrightarrow \mathrm{Mod}_S(\mathrm{Pre}(\mathscr{C}))$ by (2) where the left adjoint again is the inclusion. Since $\mathrm{Mod}_S(\mathrm{Pre}(\mathscr{C}))$ is locally presentable, and $\mathrm{Mod}_S(\mathrm{Sh}(\mathscr{C})))$ is a reflective subcategory of it, it will be locally presentable as well. From this we can conclude that it suffices to show that $\mathrm{Mod}_S(\mathbf{Sets})$ is locally presentable to conclude that the conclusion holds for arbitrary toposes as well.

Now it remains to show that $\mathrm{Mod}_S(\mathbf{Sets})$ is locally presentable, and for this we use Theorem 2.2.8. We have a forgetful functor $\mathrm{Mod}_S(\mathbf{Sets}) \to \mathbf{Sets}$, and if we show that it has a left adjoint, then it follows. This left adjoint gives the 'free algebra' on a set, and let us recall the construction of it. Write $\mathscr{L}$ for the language of S, and for an arbitrary set U we define with induction

$$X_0 = U, \quad X_{n+1} = X_n \cup \{(t, u_1, \ldots, u_{\#(t)}) \mid t \in \mathscr{L}, u_i \in X_n\}, \quad X = \bigcup_{n \in \mathbb{N}} X_n.$$

For $t \in \mathscr{L}$ and $u_1, \ldots, u_{\#(t)} \in X$ we define $t(u_1, \ldots, u_{\#(t)})$ as $(t, u_1, \ldots, u_{\#(t)})$ which makes X an algebra. Basically X consists of all terms you can make using the language and elements of U. A homomorphism from $X \to Y$ is determined by what it does on U which is shown using induction. Also, if we know a homomorphism on U, then it uniquely can be extend to one on X which shows that this is indeed a left adjoint of the inclusion.

For the last property we have a structure S with types $\mathscr{T}$ and function symbols $\mathscr{S}$. To define the finite limit structure $\mathrm{Mor}(S)$, we add for every type $T \in \mathscr{T}$ two types T and T' and a function symbol $f_T : T \to T'$ to $\mathrm{Mor}(S)$. The axioms of this structure just say that the diagram

$$\begin{array}{ccc} T_1 \times \ldots \times T_n & \xrightarrow{f} & T_0 \\ {\scriptstyle f_{T_1} \times \ldots \times f_{T_n}} \big\downarrow & & \big\downarrow {\scriptstyle f_{T_0}} \\ T_1' \times \ldots \times T_n' & \xrightarrow[f]{} & T_0 \end{array}$$

commutes for every function symbol f. This can all be written in cartesian logic, so $\mathrm{Mor}(S)$ is definable using finite limits. □

Since S-structures are defined using finite limits, they are preserved by the inverse image maps of a geometric morphisms. In addition, these preserve more formulas, namely they preserve the *geometric sequents*.

Definition 6.1.5 (Geometric Sequent)**.** Let $\mathscr{L}$ be a language with types $\mathscr{T}$, function symbols $\mathscr{F}$ and relation symbols $\mathscr{R}$. Then a *geometric sequent* of $\mathscr{L}$ is a formula of the form $\forall_{x_1} \ldots \forall_{x_n} [\varphi \Rightarrow \psi]$ where φ and ψ are built from $\exists$, finite conjunctions and infinite disjunction.

In this language one can say that a map f is a monomorphism with the formula $\forall_x \forall_y [f(x) = f(y) \Rightarrow x = y]$. With this terminology we can state and prove the following proposition.

Proposition 6.1.6. *Given a structure S defined in terms of finite limits, a topos $\mathscr{E}$, and a set $\mathscr{A}$ consisting of geometric sentences in the language of S. Then we have*

(1) *If $\mathscr{E} \to \mathscr{F}$ is a geometric morphism and $X \in \mathrm{Mod}_S(\mathscr{F})$ such that $X \models \mathscr{A}$, then $f^*(X) \models \mathscr{A}$.*
(2) *$\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E})$ is closed under filtered colimits.*
(3) *Given a small category $\mathscr{D}$ and for an object $d \in \mathscr{D}$ define the functor $\varepsilon_d : \mathscr{E}^{\mathscr{D}} \to \mathscr{E}$ which evaluates a functor at d. Then for $X \in \mathrm{Mod}_S(\mathscr{E}^{\mathscr{D}})$ we have $X \models \mathscr{A}$ iff for all objects d of $\mathscr{D}$ we have $\varepsilon_d(X) \models \mathscr{A}$.*
(4) *The category $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E})$ is accessible, and the inclusion $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E}) \to \mathrm{Mod}_S(\mathscr{E})$ is accessible.*

PROOF. Because geometric morphisms preserve S-structures and geometric formulas, (1) follows immediately. To show that such a geometric morphism preserves formulas, we just need that it commutes with finite limits and arbitrary colimits, and filtered colimits enjoy that property as well, so (2) follows.

Since evaluation at d preserves finite limits and arbitrary colimits, it thus preserves geometric formulas. Also, limits and colimits in $\mathscr{E}^{\mathscr{D}}$ are evaluated pointwise, so to determine the truth of a formula in $\mathscr{E}$ we have to check it at every point. Hence, we have $X \models \mathscr{A}$ iff for all objects d we have $\varepsilon_D(X) \models \mathscr{A}$. Now note that (3) says that $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E}^{\mathscr{D}})$ and $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E})^{\mathscr{D}}$ are isomorphic.

Lastly, we show (4), and for this it suffices to show that $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E})$ is accessible, because then it is accessibly embedded in $\mathrm{Mod}_S(\mathscr{E})$. We start by showing that $\mathrm{Mod}_{S,\mathscr{A}}(\mathbf{Sets})$ is accessible. For this we use the downward Löwenheim-Skolem Theorem for which we refer the reader to Theorem 2.3.7 in [**Mar02**]. Note that the category $\mathrm{Mod}_{S,\mathscr{A}}(\mathbf{Sets})$ has all filtered colimits by part ii. Let κ be the cardinality of $\mathscr{T} \cup \mathscr{F} \cup \mathscr{R}$, so it is the total number of symbols in the language of S, and let I be the set of all S-structures which satisfy every sentence in $\mathscr{A}$ and have cardinality at most κ. If we have an S-structure X, then we can write X as a colimit of elements of I. For x in X we can find an elementary substructure Y_x of X with cardinality at most κ by the downward Löwenheim-Skolem Theorem. Note that Y_x satisfies all axioms in $\mathscr{A}$, because it is an elementary substructure. Then we have $X = \mathrm{colim}_{x \in X} Y_x$, and thus $\mathrm{Mod}_{S,\mathscr{A}}(\mathbf{Sets})$ is indeed accessible.

Since the Löwenheim-Skolem Theorem holds in arbitrary Grothendieck toposes [**ack**], this argument can be applied to arbitrary Grothendieck toposes. Hence, all $\mathrm{Mod}_{S,\mathscr{A}}(\mathscr{E})$ are accessible. □

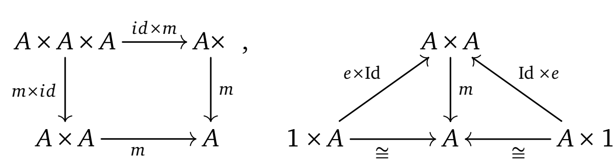
For the last proposition we look at geometric morphisms. Obviously, if a functor preserves colimits and finite limits, then it preserves the truth of geometric formulas. However, less obviously, if a geometric morphism is surjective as well, then it reflects the truth of geometric formulas.

Proposition 6.1.7. *Let $f^* \dashv f_*$ be a surjective geometric morphism. Then we have $A \leq B$ iff $f^*(A) \leq f^*(B)$.*

PROOF. Note that surjective means that f^* is faithful, and thus it is injective on subobjects. Also, if $A \leq B$, then $f^*(A) \leq f^*(B)$, because f^* preserves all colimits and finite limits. Now suppose that $f^*(A) \leq f^*(B)$ where A and B are subobjects of E. Note that $f^*(A \wedge B) = f^*(A) \wedge f^*(B) = f^*(A)$, because f^* preserves finite limits and $f^*(A) \leq f^*(B)$. Because f^* is injective on subobjects, we have $A = A \wedge B$, and from this we can conclude that $A \leq B$. □

6.2. Sketches and Definable Functors

Another way to define 'definable' is via *sketches*. When we use sketches, we will use diagrams to define the structure. Before diving into formal definitions, let us look at an example first. A monoid object in a category is an object A with arrows $m : A \times A \to A$ and $e : 1 \to A$ such that the following diagrams commute



To formulate such definitions in general, we need to say that some arrows exist and that some diagrams commute. However, we also need to say that the domain of the arrow is a product, and in more general cases we might want to use more general limits or colimits as well. To do this, we use sketches.

Definition 6.2.1 (Sketch). A *sketch* $\mathbb{S}$ consists of a diagram $\mathscr{D}$ with a set U of cones on $\mathscr{D}$ and a set V of cocones on $\mathscr{D}$. A *model* of a sketch in a category $\mathscr{C}$ is a functor $\mathscr{D} \to \mathscr{C}$ which sends the cones in U to limiting cones and the cocones in $\mathscr{D}$ to colimiting cocones. We define $\mathbb{S}[\mathscr{C}]$ to be the full subcategory of $\mathscr{C}^{\mathscr{D}}$ of models of the sketch $\mathbb{S}$.

More concretely, what is a sketch? Let us consider the example for a group, and for this we first need to describe a category. We need an arrow from $m : B \to A$, an arrow $i : A \to A$ and $e : C \to A$, and we need to say $B = A \times A$ and $C = 1$. This means that C is a limiting cone and that we have two arrows $\pi_1, \pi_2 : B \to A$ such that the following diagram is a limiting cone

$$A \xleftarrow{\pi_1} B \xrightarrow{\pi_2} A$$

Now we have the required structure, but we also need to guarantee that certain diagrams commute. We will only show how to say that $C \to A$ is a unit for m, because the other diagrams can be described in the same way. We need two auxiliary objects E and E' which represent $A \times C$ and $C \times A$ respectively, and for this we need two cones. They also have arrows $p_1 : E \to A$, $p_2 : E \to 1$, $q_1 : E' \to 1$ and $q_2 : E' \to A$. The other ingredient are the product arrows and for this we need two diagrams

$$\begin{array}{ccccc} A & \xleftarrow{p_1} & E & \xrightarrow{p_2} & C \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle e} \\ A & \xleftarrow[\pi_1]{} & B & \xrightarrow[\pi_2]{} & A \end{array} \qquad \begin{array}{ccccc} C & \xleftarrow{q_1} & E' & \xrightarrow{q_2} & A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle \mathrm{Id}} \\ A & \xleftarrow[\pi_1]{} & B & \xrightarrow[\pi_2]{} & A \end{array}$$

Now we require that the following diagram commutes

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle r_1}\nearrow & \downarrow{\scriptstyle m} & \nwarrow{\scriptstyle r_2} & \\ E & \xrightarrow[p_1]{} & A & \xleftarrow[q_2]{} & E' \end{array}$$

This is precisely the unit law for monoids. In a similar fashion we can also state that m is associative, and combining all this stuff we get a category $\mathscr{D}$ and cones such that the models of this sketches are precisely the monoids.

Geometric formulas are defined using finite limits and arbitrary colimits, and similarly we can define geometric sketches. If all cones in some sketch are finite, then that sketch is called geometric. Universal algebras can be defined using geometric sketches in the same way as we defined monoids. We start by defining the maps, and since every map has a finite arity, we only need to consider finite products. Hence, only finite cones are needed to say that the maps have the right domain. To show the required laws, we again only need to consider finite products and since all these laws are equational, we can formulate it by stating that certain diagrams commute. Hence, universal algebras can be defined using geometric sketches.

One important application of sketches is that it allows us to define definable functors. For that we need some extra terminology. Given two sketches $\mathbb{S}_1 = (\mathscr{D}_1, U_1, V_1)$ and $\mathbb{S}_2 = (\mathscr{D}_2, U_2, V_2)$, a *sketch morphism* is a functor $F : \mathscr{D}_1 \to \mathscr{D}_2$ which sends cones of U_1 to U_2 and cocones of V_1 to V_2. This gives a functor from $\mathbb{S}_2[\mathscr{C}]$ to $\mathbb{S}_1[\mathscr{C}]$ by precomposition: a functor $G : \mathscr{D}_2 \to \mathscr{C}$ is sent to $G \circ F$. A sketch morphism is called *rigid* if for every topos $\mathscr{E}$ this induced morphism is an equivalence.

The main issue for defining definable functors is that we want to have a functor for the models in every category. To solve that, we want that the graph is definable, and then the model can be used to give an actual functor for arbitrary interpretations.

Definition 6.2.2 (Definable Functor)**.** Let two sketches $\mathbb{S}_1$ and $\mathbb{S}_2$ be given. Then a *definable functor* from $\mathbb{S}_1$ to $\mathbb{S}_2$ is a sketch $\mathbb{G}$, the graph, with sketch morphism $p_1 : \mathbb{S}_1 \to \mathbb{G}$ and $p_2 : \mathbb{S}_2 \to \mathbb{G}$ such that p_1 is rigid. We say that it is *geometrically definable* if $\mathbb{G}$ is a geometric sketch.

Since a sketch morphism $\mathbb{S} \to \mathbb{T}$ gives a morphism $\mathscr{T}[\mathbb{C}] \to \mathscr{S}[\mathbb{C}]$, the arrows p_1 and p_2 are in the order opposite to which you would expect. To say that every point has precisely one image, we say that p_1 is rigid. For a topos $\mathscr{E}$ the map $p_1 : \mathscr{E}[\mathbb{G}] \to \mathscr{E}[\mathbb{S}_1]$ has a quasi-inverse $s : \mathscr{E}[\mathbb{S}_1] \to \mathscr{E}[\mathbb{G}]$, so we get a functor $\mathscr{E}[\mathbb{S}_1] \to \mathscr{E}[\mathbb{G}] \to \mathscr{E}[\mathbb{S}_2]$ by composition.

Let us now consider an example of a definable functor, namely the free algebra functor. To do this, we need to describe this functor in a slightly different way, and that construction can be described using sketches.

Example 6.2.3. In the proof of Proposition 6.1.4 we defined a functor T which assigns to every set U the free algebra generated by U. To show that this functor is definable, we need to make a sketch for its graph. The main point here is that the free algebra on U can be described as a colimit. If $\mathbb{D}$ is the category with objects (α, x) where α is a finite ordinal and $x \in T(\alpha)$, and arrows from (α, x) to (β, y) are functions $f : \alpha \to \beta$ such that $T(f)(\alpha) = \beta$, then we can define a diagram $F : \mathbb{D} \to \mathscr{E}$ sending (α, x) to U^α. Here U^α is the α-fold product of U.

To define the structure maps, note that the diagram $\mathscr{D}$ is ω-filtered. Since ω-filtered colimits commute with finite limits, we can make the structure map of each function symbol f of arity n by mapping an n-tuple $((\alpha_1, x_1), \ldots, (\alpha_n, x_n))$ into some (β, y). Without loss of generality we can assume $\alpha_1 \geq \alpha_i$ for all i. Now we have inclusions $\alpha_i \subseteq \alpha_1$, and we define β to be α_1 and y to be $f(x_1, \ldots, x_n)$. With this definition all equations of the algebra are satisfied, because they hold for $T(\alpha)$. One can show now that $\operatorname{colim} F$ is the free T-algebra on U.

All of this can be summarized in a countable sketch, and thus the free algebra functor is definable using a geometric sketch. Note that it is left adjoint to the forgetful functor which can be defined using finite limits.

Lastly, we need three extra propositions which give some properties of definable functors and sketches. The first property is similar to (1) of Proposition 6.1.4: it just says that the category of models of some finite limit sketch is locally presentable.

Proposition 6.2.4. *Let $\mathbb{S}$ be a sketch defined using finite limits. Then for every topos $\mathscr{E}$ the category $\mathbb{S}[\mathscr{E}]$ is locally presentable.*

The second property is says that all functors, which are between finite limit sketches and defined using finite limits, have a definable adjoint. The adjoint, however, can be defined using a geometric sketch instead of a finite limit sketch.

Proposition 6.2.5. *Given are sketches $\mathbb{S}_1$ and $\mathbb{S}_2$ which are defined using finite limits, and a definable functor R from $\mathbb{S}_1$ to $\mathbb{S}_2$ which also is defined using finite limits. Then* *there is a geometrically definable functor L from $\mathbb{S}_2$ to $\mathbb{S}_1$ such that for every topos $\mathscr{E}$ we have $L_{\mathscr{E}} \dashv R_{\mathscr{E}}$ where $L_{\mathscr{E}} : \mathbb{S}_1[\mathscr{E}] \to \mathbb{S}_2[\mathscr{E}]$ and $R_{\mathscr{E}} : \mathbb{S}_2[\mathscr{E}] \to \mathbb{S}_1[\mathscr{E}]$ are induced by L and R respectively.*

We will not prove this in detail, and for a detailed proof we refer the reader to [**Bek01**]. Instead we will give the main ingredients of the proof. The main idea of the proof is to use the *classifying topos of a geometric theory*, and for more details on this we refer the reader to [**MLM92**]. The classifying topos $\mathscr{C}_T$ of a theory T is a topos such that for every topos $\mathscr{E}$ geometric morphisms from $\mathscr{E}$ to $\mathscr{C}_T$ correspond with T-models of $\mathscr{E}$. All geometric theories have a classifying topos, and thus theories defined using finite limits as well. The theorem we need to prove this statement, says that a geometric sequent holds in every model of T iff it holds in the universal model U_T in the classifying topos. So, if we can find a classifying topos, then we can find a universal model, and to check whether a statement holds for all models, it is sufficient to check that it holds for the universal example. To prove Proposition 6.2.5, one checks that it holds for the universal example, and then one can extend it to the general case.

The last property just says that definable functors between toposes preserves limits and filtered colimits.

Proposition 6.2.6. *Let sketches $\mathbb{S}_1$, $\mathbb{S}_2$ and let a definable functor F from S_1 to S_2 be given such that S_1, S_2 and F are all defined using finite limits. Then for all toposes $\mathscr{E}$ the induced functor $R_{\mathscr{E}} : \mathbb{S}_1[\mathscr{E}] \to \mathbb{S}_2[\mathscr{E}]$ preserves filtered colimits and limits.*

Proof. By definition F has a graph $\mathbb{G}$ and we have sketch morphisms $p_1 : \mathbb{S}_1 \to \mathbb{G}$ and $p_2 : \mathbb{S}_2 \to \mathbb{G}$ with p_1 rigid. It suffices to show that for sketches $\mathbb{S}_1$ and $\mathbb{S}_2$ with a sketch morphism $m : \mathbb{S}_1 \to \mathbb{S}_2$ the functor $\mathbb{S}_2[\mathscr{E}] \to \mathbb{S}_1[\mathscr{E}]$ preserves filtered colimits and limits. Note that p_1 and p_2 induce functors $\widehat{p_1} : \mathbb{G}[\mathscr{E}] \to \mathbb{S}_1[\mathscr{E}]$ and $\widehat{p_2} : \mathbb{G}[\mathscr{E}] \to \mathbb{S}_2[\mathscr{E}]$, and the functor $\widehat{p_1}$ has a quasi-inverse s. Now F is defined as $\widehat{p_2} \circ s$, and if both $\widehat{p_2}$ and s preserve limits and filtered colimits, then we are done. To show that s preserve limits and filtered colimits, it suffices to show that $\widehat{p_1}$ preserve filtered colimits and limits because

$$\lim s(X_i) \cong s(\widehat{p_1}(\lim s(X_i))) \cong s(\lim \widehat{p_1}(s(X_i))) \cong s(\lim X_i).$$

The next step is thus to show that for a sketch morphism $m : \mathbb{S}_1 \to \mathbb{S}_2$ the induced map $\widehat{m} : \mathbb{S}_2[\mathscr{E}] \to \mathbb{S}_1[\mathscr{E}]$ preserves limits and colimits. Write $\mathbb{D}_1$ and $\mathbb{D}_2$ for the diagrams of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. Since the sketch morphism gives a map $\mathbb{D}_1 \to \mathbb{D}_2$, we can form the following commutative diagram

$$\begin{array}{ccc} \mathscr{E}^{\mathbb{D}_2} & \xrightarrow{m^*} & \mathscr{E}^{\mathbb{D}_1} \\ \uparrow & & \uparrow \\ \mathbb{S}_2[\mathscr{E}] & \xrightarrow[m]{} & \mathbb{S}_1[\mathscr{E}] \end{array}$$

Since limits and colimits in $\mathscr{E}^{\mathbb{D}_1}$ and $\mathscr{E}^{\mathbb{D}_2}$ are evaluated pointwise, the functor m^* preserves all limits and colimits. Also, the inclusion $\mathbb{S}_2[\mathscr{E}] \to \mathscr{E}^{\mathbb{D}_2}$ preserves all limits and colimits. If we show that the inclusion $\mathbb{S}_1[\mathscr{E}] \to \mathscr{E}^{\mathbb{D}_1}$ reflects all limits and filitered colimits, then we are done.

So, now we are dealing with the inclusion functor $\mathbb{S}_1[\mathscr{E}] \to \mathscr{E}^{\mathbb{D}_1}$, and we want to show this functor has a left adjoint. For this we start by noting that $\mathscr{E}^{\mathbb{D}_1}$ is definable using finite limits and colimits, because we just use the diagram D_1 with no further cones or cocones. Note that the inclusion functor is definable, and hence from Proposition 6.2.5 follows that it has a left adjoint. □

CHAPTER 7

Sheafifying Model Structures

In Chapter 3 we constructed some model structures on some categories using Quillen's small object argument and transfer. This way one can also define a model structure on simplicial objects in a topos [**Joy83**]. However, under suitable assumptions this happens automatically. The main point of [**Bek00, Bek01**] is that if we have a model structure on structured sets, then we also get a model structure with the same definitions on structured sheaves under suitable assumptions. One of these assumptions says that the cofibrations and weak equivalences should be defined with geometric formulas. This is used, because then we can use Boolean localization from Theorem 5.4.1 to show certain formulas.

For the remaining axioms we will need a more powerful tool which we discuss in Section 7.1. This is another variant of the tools discussed in Section 3.1, but this time the solution set condition is crucial. Using Proposition 2.2.12 we can check for accessibility to show this condition, and then using Propositions 6.1.4 and 6.1.6 we can conclude.

7.1. A Theorem by Jeff Smith

In this section we discuss another theorem which we can use to detect model structures. The important thing about it is that one of the conditions is the solution set condition. For the proof we need to construct the generating trivial cofibrations, and they can be constructed by using the solution set condition.

Theorem 7.1.1. *Let $\mathscr{C}$ be a locally presentable category, let* We *be a subcategory, and let I be a set of morphisms of $\mathscr{C}$. Suppose the following*

(1) We *is closed under retracts and satisfies the 2-out-of-3 property;*
(2) *We have* $\mathrm{Inj}(I) \subseteq \mathrm{We}$*;*
(3) $\mathrm{Cof}(I) \cap \mathrm{We}$ *is closed under transfinite composition and pushout;*
(4) We *satisfies the solution set condition at I.*

Then we have a combinatorial model structure on $\mathscr{C}$ where the weak equivalences are We*, the cofibrations are* $\mathrm{Cof}(I)$ *and the fibrations are* $\mathrm{Inj}(\mathrm{Cof}(I) \cap \mathrm{We})$*.*

To prove Theorem 7.1.1 we use Theorem 3.1.4. Note that (1) and (3) in Theorem 3.1.4 hold by assumption, and thus the only remaning thing is to find a generating set for $\mathrm{Cof}(I) \cap We$. We will do this in an indirect way by constructing a set J which satisfies some solution set condition which then turns out to generate the trivial cofibrations. For this, let us introduce some temporary terminology. Call a set $J \subseteq \mathrm{Cof}\, I \cap We$ *nice* if every square

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle w} \\ Y & \longrightarrow & B \end{array}$$

with $i \in I$ and $w \in We$ can be factored as

$$\begin{array}{ccccc} X & \longrightarrow & E & \longrightarrow & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle w} \\ Y & \longrightarrow & F & \longrightarrow & B \end{array}$$

where $j \in J$. The reason why we are interested in nice sets is given by the following two lemmas.

Lemma 7.1.2. *If J is nice, then we can factor $f \in \mathrm{We}$ as $h \circ g$ with $h \in \mathrm{Inj}(I)$ and $g \in \mathrm{Cell}(J)$.*

PROOF. Here we imitate the proof of Quillen's Small Object Argument Theorem 3.1.2. Again note that we can find an ordinal number λ such that every domain X_i of a map in I is λ-small. Define X_0 to be X.

Now suppose we have an ordinal number α and X_α with $h_\alpha : X_\alpha \to Y$. Consider all diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & X_\alpha \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h_\alpha} \\ B & \xrightarrow[g']{} & Y \end{array}$$

Because J is nice, we can find a $j_{(i,g,g')} \in J$ and a factorization

$$\begin{array}{ccccc} A & \longrightarrow & C_{(i,g,g')} & \longrightarrow & X_\alpha \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (i,g,g')} & & \downarrow{\scriptstyle h_\alpha} \\ B & \longrightarrow & D_{(i,g,g')} & \longrightarrow & Y \end{array}$$

Now we can form coproducts acquiring the following square

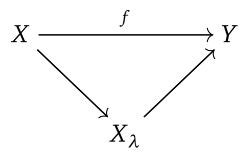
$$\begin{array}{ccc} \coprod_{(i,g,g')} C_{(i,g,g')} & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow{\scriptstyle h_\alpha} \\ \coprod_{(i,g,g')} D_{(i,g,g')} & \longrightarrow & Y \end{array}$$

Again the maps are defined in a similar fashion as in Theorem 3.1.2, and we form the pushout

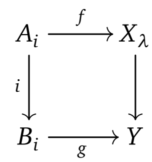
$$\begin{array}{cccc} \coprod_{(i,g,g')} C_{(i,g,g')} & \longrightarrow & X_\alpha & \\ \downarrow & & \downarrow{\scriptstyle h_\alpha} & \searrow \\ \coprod_{(i,g,g')} D_{(i,g,g')} & \longrightarrow & P & \\ & \searrow & & \searrow \\ & & & Y \end{array}$$

We define $X_{\alpha+1}$ to be P.

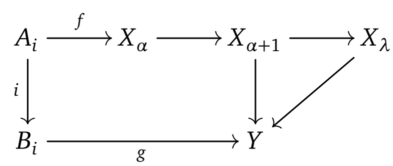
Lastly, for a limit ordinal α we define X_α to be the colimit of X_β for $\beta < \alpha$. Now we can factorize f as follows



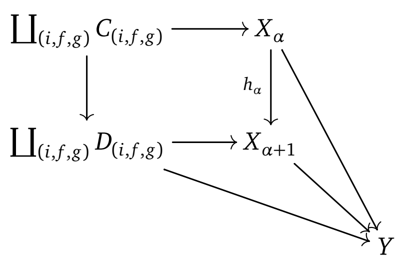
By construction the map $X \to X_\lambda$ is a J-cellular complex, so it remains to show that the map $X_\lambda \to Y$ is I-injective. So, suppose we have the following diagram



Since A_i is small, we can factorize the map $A_i \to X_\lambda$ through some X_α with $\alpha < \lambda$ and then we get



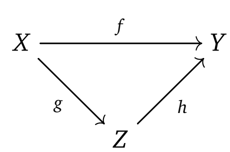
By construction we have the following pushout square



We have maps from A_i into $\coprod_{(i,f,g)} C_{(i,f,g)}$ and B_i into $\coprod_{(i,f,g)} D_{(i,f,g)}$ by construction. We factorized the map $A_i \to X_\alpha$ via $C_{(i,f,g)}$, and we did the same for the map from $B_i \to Y$. This gives the required lift $B_i \to X_{\alpha+1}$, and this concludes the proof of this lemma. □

Lemma 7.1.3. *If J is nice, then we have* $\mathrm{Cof}(J) = \mathrm{Cof}(I) \cap \mathrm{We}$.

PROOF. Since $\mathrm{Cof}(I) \cap We$ is closed under transfinite composition and pushout and $J \subseteq \mathrm{Cof}(I) \cap We$, we have that $\mathrm{Cof}(J) \subseteq \mathrm{Cof}(I) \cap We$. Conversely, if we have $f \in \mathrm{Cof}(I) \cap We$, we can factorize it as



where $h \in \mathrm{Inj}(I)$ and $g \in \mathrm{Cell}(J)$. We want to show that $f \in \mathrm{Cof}(I)$, and we do that by showing that f is a retract of g. Consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Y & = & Y \end{array}$$

Since $f \in \mathrm{Cof} I \cap We$ and $h \in \mathrm{Inj}(I)$, we have a lift $Y \to Z$ and this shows that f is a retract of g. □

Hence, if we can find a nice set J, then it generates the trivial cofibrations, and this will allow us to conclude the argument. To define J, let us consider all squares of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle w} \\ B & \longrightarrow & Y \end{array}$$

where $i \in I$ and $w \in We$. Because We satisfies the solution condition at I, we can factor

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle w_i} & & \downarrow{\scriptstyle w} \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

where $w_i \in W$. Our goal is to find a factorization for another square, namely the following

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle w_i} \\ B & \longrightarrow & D \end{array}$$

First, we form the following pushout P

$$\begin{array}{cccc} A & \longrightarrow & C & \\ {\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & \searrow{\scriptstyle w_i} \\ B & \longrightarrow & P & \\ & \searrow & & \searrow{\scriptstyle h} \\ & & & D \end{array}$$

and note that we get $h : P \to D$ by the universal property of the pushout. Now we factorize h as $q \circ p$ where $q : Q \to D$ with $p \in \mathrm{Cof}(I)$ and $q \in \mathrm{Inj}(I)$, so q is a weak equivalence. By the 2-out-of-3 property we then have $p \circ i' \in We$, and we define J to be set set of all $p \circ i'$ taking one for each w_i. Now we can factorize our original square as follows

$$\begin{array}{ccccccccc} A & & \longrightarrow & & C & & \longrightarrow & & X \\ {\scriptstyle i}\downarrow & & & & \downarrow{\scriptstyle p \circ i'} & & & & \downarrow{\scriptstyle w} \\ B & \longrightarrow & P & \longrightarrow & Q & \xrightarrow[q]{} & D & \longrightarrow & Y \end{array}$$

Hence, J is indeed a nice set, and from this we conclude Theorem 7.1.1.

7.2. Sheafifying Homotopy

Let us start with some arbitrary topos $\mathscr{E}$. As discussed before, toposes $\mathrm{Sh}(\mathscr{B})$ for complete Boolean algebras $\mathscr{B}$ are useful to determine the truth of formulas in an arbitrary topos. Since we have Boolean localization, we can find a surjective geometric morphism $\mathrm{Sh}(\mathscr{B}) \to \mathscr{E}$. This adjunction preserves and reflects the truth of geometric formulas, and thus to show that a geometric formula holds in every topos, it suffices to show that it holds in all $\mathrm{Sh}(\mathscr{B})$. For this reason Boolean localization is a useful tool in topos theory, because it allows us to check something for arbitrary toposes by just checking it for certain toposes.

On the other hand, there are toposes for which this is simpler, and there are toposes with *enough points*. A *point* of a topos is a geometric morphism $\mathbf{Sets} \to \mathscr{E}$. This definition is easy to understand, because a point of a topological space X is a map $* \to X$ which gives a geometric morphism $\mathbf{Sets} \to \mathrm{Sh}(X)$. We say that a topos $\mathscr{E}$ has *enough points* iff for every map $f : A \to B$ in $\mathscr{E}$ we have that f is an isomorphism if for every point point $p : \mathbf{Sets} \to \mathscr{E}$ the map $p^*(f)$ is an isomorphism. It can be shown that a geometric formula holds in a topos with enough points iff it holds in **Sets**. For these toposes it is easier, but our focus will be the general case.

This bring us to the following idea: if we have a model structure on ‘structured sets’, can we turn it into a model structure on ‘structured’ sheaves? More concisely, can we sheafify model structures? For example, we have a model structure on simplicial sets which are the simplicial objects in **Sets**. Can we use this to find a model structure for simplicial objects in arbitrary toposes? Also, we have a model structure on chain complexes, and the question is whether we can use it to find a model structure for chain complexes of sheaves of abelian groups. In general there is no reason why this would be true. However, if the model structure is defined using a language which is preserved by the geometric morphisms, then the answer to the question is yes under a mild condition. The condition says that for every topos the cofibrations must be generated by some set which holds whenever the cofibrations are the monomorphisms.

In [**Bek00**] a theorem is discussed which answers this question, and using the material discussed until now, we can readily prove the theorem. The main idea is that we want to apply Theorem 7.1.1 to arbitrary toposes. Since the cofibrations might not be the monomorphisms, we will need that for every topos the cofibrations are generated by some set. Because *We* is defined using geometric sentences, the category of its models is locally presentable, and thus accessible. Therefore, the solution set condition is satisfied, and this solves one of the main difficulties. Now we also assume that we have a model structure in some elementary cases (like **Sets** or $\mathrm{Sh}(\mathscr{B})$) from which we can transfer it to arbitrary toposes. It is also not difficult to check (3), because both the cofibrations and the weak equivalences are closed under transfinite composition and pushout. Only (2) requires some work where we do it in two steps. First, we extend it from **Sets** to presheaves, and then we solve it for arbitrary toposes by using logical methods. For presheaves we can do it by hand, but for arbitrary toposes we can use logical methods. Recall that $\mathrm{Mod}_{S,W}(\mathscr{E})$ consists of all S structures which satisfy all axioms in W.

Theorem 7.2.1. *Let S be a structure defined with finite limits, and let W and C be collections of geometric sentences.*

(1) $\mathrm{Mod}_S(\mathrm{Sh}(\boldsymbol{Sets}))$ *is a model category with weak equivalences* $\mathrm{Mod}_{S,W}(\boldsymbol{Sets})$ *and cofibrations* $\mathrm{Mod}_{S,C}(\boldsymbol{Sets})$.

(2) $\mathrm{Mod}_S(\mathrm{Sh}(\mathscr{B}))$ *is a model category with weak equivalences* $\mathrm{Mod}_{S,W}(\mathscr{B})$ *and cofibrations* $\mathrm{Mod}_{S,C}(\mathscr{B})$ *where* $\mathscr{B}$ *is a complete Boolean algebra.*
(3) *For every topos* $\mathscr{E}$ *there is a set* $I_{\mathscr{E}}$ *such that* $\mathrm{Mod}_{S,C}(\mathscr{E}) = \mathrm{Cof}(I_{\mathscr{E}})$.

Then for every topos $\mathscr{E}$ *we have a model category* $\mathrm{Mod}_S(\mathscr{E})$ *with weak equivalences* $\mathrm{Mod}_{S,W}(\mathscr{R})$ *and cofibrations* $\mathrm{Mod}_{S,C}(\mathscr{E})$.

PROOF. We apply Theorem 7.1.1. Note that toposes are locally presentable, so we can apply it. The fourth condition follows from (4) in Proposition 6.1.6 and Proposition 2.2.12. With Boolean localization we can show the first property. We need to check a statement of the form

$$\forall_f \forall_g [W(f) \wedge W(g) \Rightarrow W(g \circ f)]$$

where $W(f)$ and $W(g)$ are geometric, and thus this condition is a geometric sentence. Because it holds in all toposes $\mathrm{Sh}(\mathscr{B})$, we can conclude with Boolean localization that it holds in all toposes. Also, $\mathrm{Mod}_{S,W}(\mathscr{E})$ is closed under colimits by (2) in Proposition 6.1.6, so if we show that retracts are colimits, then we are done. Let $\mathscr{C}$ be the category with one object $*$ and an arrow $g : * \to *$ satisfying $g \circ g = \mathrm{Id}$. The colimit of the diagram F which maps g to f, is a retract of f.

Next we show that (3) of Theorem 7.1.1 holds as well. Being a pushout is a geometric property, so to check it we need to check a statement of the form

$$\forall_f \forall_g \forall h [W(f) \wedge C(f) \wedge g \text{ is a pushout of } f \Rightarrow W(g) \wedge C(g)]$$

So again it follows from Boolean localization. To show that $\mathrm{Mod}_{S,W}(\mathscr{E}) \cap \mathrm{Mod}_{S,C}(\mathscr{E})$ is closed under transfinite composition, it suffices by (2) of Proposition 6.1.6 to show that it is closed under composition. But this is a geometric statement, because it is of the form

$$\forall_f \forall_g [W(f) \wedge C(f) \wedge W(g) \wedge C(g) \Rightarrow W(g \circ f) \wedge C(g \circ f)].$$

Hence, it follows from Boolean localization.

Lastly, we show condition (2) of Theorem 7.1.1. For this we need two steps: first we show it for preshaves, and then we show it for toposes. For presheaves we use the fact that everything is done pointwise, and for toposes we use sheafification. First note that (2) holds for **Sets**, because of the first assumption.

Now consider a presheaf category $\mathscr{E} = \mathbf{Sets}^{\mathbb{D}^{\mathrm{op}}}$ on a small category $\mathbb{D}$, and any arrow $f : X \to Y$ which is in $\mathrm{Inj}(\mathrm{Mod}_{S,C}(\mathscr{E}))$. Since evaluation is defined by left Kan extension, it has a left adjoint L. To check that f satisfies all sentences in W, we need to check at every object d of $\mathbb{D}$ that the evaluation $\varepsilon_d(f)$ of f at d satisfies W. Thus we need to show that $\varepsilon_d(f) \in \mathrm{Mod}_{S,W}(\mathbf{Sets})$, and because (2) holds in **Sets**, it suffices to show that $\varepsilon_d(f) \in \mathrm{Inj}(I_{\mathbf{Sets}})$. To show that $\varepsilon_d(f) \in \mathrm{Inj}(I_{\mathbf{Sets}})$, we show that $f \in \mathrm{Inj}(L(\mathrm{Mod}_{S,C}(\mathbf{Sets})))$. So, suppose that we have shown that $f \in \mathrm{Inj}(L(\mathrm{Mod}_{S,C}(\mathbf{Sets})))$, and consider the square

$$\begin{array}{ccc} A & \longrightarrow & X_d \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \varepsilon_d(f)} \\ C & \longrightarrow & Y_d \end{array}$$

where $h \in I_{\mathbf{Sets}}$. By adjunction we can factor this diagram as follows

$$\begin{array}{ccccc} A & \longrightarrow & L(A)_d & \longrightarrow & X_d \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle L(h)} & & \downarrow{\scriptstyle \varepsilon_d(f)} \\ C & \longrightarrow & L(C)_d & \longrightarrow & Y_d \end{array}$$

Since $h \in I_{\mathbf{Sets}}$ and $f \in \mathrm{Inj}(L(\mathrm{Mod}_{S,C}(\mathbf{Sets})))$ by assumption, we can find a lift. Hence, it suffices to show that $f \in \mathrm{Inj}(L(\mathrm{Mod}_{S,C}(\mathbf{Sets})))$.

To show that $f \in \mathrm{Inj}(L(\mathrm{Mod}_{S,C}(\mathbf{Sets})))$, we show that for $g \in \mathrm{Mod}_{S,C}(\mathbf{Sets})$ we have $L(g) \in \mathrm{Mod}_{S,C}\,\mathscr{E}$. And for that we again evaluate objectwise, meaning that we have to show for objects d of $\mathbb{D}$ that $L(g)_d \in \mathrm{Mod}_{S,C}(\mathbf{Sets})$. By definition of the left Kan extension all these $L(g)$ are copowers of g and copowers of cofibrations are cofibrations. Therefore, $L(g)$ is indeed a cofibration between presheaves from which we can conclude (2) for presheaves.

Lastly we show that now we can conclude it for all toposes. Let $\mathscr{E} = \mathrm{Sh}(\mathbb{D})$ be a topos and let $\mathscr{F} = \mathbf{Sets}^{\mathbb{D}^{\mathrm{op}}}$. Also, take a map $f \in \mathrm{Inj}(\mathrm{Mod}_{S,C}(\mathscr{E}))$. Our goal is to show that $f \in \mathrm{Mod}_{S,W}(\mathscr{E})$, and for this we use sheafification. Note that we have a left exact left adjoint $a : \mathscr{F} \to \mathscr{E}$ given by sheafification, and that note that $f \cong a(i(f))$. Because sheafification preserves geometric formulas, it thus suffices to show that $i(f)$ is a weak equivalence. Since we know (2) holds for presheaf categories, it thus suffices to show that $i(f) \in \mathrm{Inj}(\mathrm{Mod}_{S,C}(\mathscr{F}))$. So, consider the following square with $h \in \mathrm{Mod}_{S,C}(\mathscr{F})$

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i(f)} \\ B & \longrightarrow & Y \end{array}$$

Let us sheafifiy the diagram to obtain

$$\begin{array}{ccccc} A & \longrightarrow & a(A) & \longrightarrow & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle a(h)} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & a(B) & \longrightarrow & Y \end{array}$$

Since a preserves the class of cofibrations, the right square has a lift which gives the desired lift. □

In the next section we will discuss concrete examples of this theorem, but now we look at a second version. The next theorem says that we can do the same with transfer. However, for this we need the language of sketches and definable functors from Section 6.2. This is because we would like to have a description of the functor for every topos, and a sketches precisely give that. For this reason also the structures need to be defined using sketches, because we need that to talk about definable functors. Now we want to apply transfer on every topos, so in this case for one of the structures we already must have a model structure. To test whether the conditions of Proposition 3.2.1 hold, we need that they hold for all sheaf toposes on complete Boolean algebras.

Theorem 7.2.2. *Given are two sketches $\mathbb{S}_1$ and $\mathbb{S}_2$ defined using finite limits, and a definable functor R from $\mathbb{S}_2$-structures to $\mathbb{S}_1$-structures defined using finite limits. Also, let W and C be collections of geometric sentences in the language of morphisms of S_1 structures. Suppose that*

(1) *For every topos $\mathscr{E}$ we have a combinatorial model structure on $\mathbb{S}_1[\mathscr{E}]$ with weak equivalences* $\mathrm{Mod}_{S_1,W}(\mathscr{E})$ *and cofibrations* $\mathrm{Mod}_{S_1,C}(\mathscr{E})$.
(2) *For every complete Boolean algebra $\mathscr{B}$ the functor $\mathbb{S}_2[\mathrm{Sh}(\mathscr{B})] \to \mathbb{S}_1[\mathrm{Sh}(\mathscr{B})]$ creates a model structure on $\mathbb{S}_2[\mathrm{Sh}(\mathscr{B})]$.*

Then for every topos $\mathscr{E}$ the functor $\mathbb{S}_2[\mathscr{E}] \to \mathbb{S}_1[\mathscr{E}]$ creates a model structure on $\mathbb{S}_2[\mathscr{E}]$.

PROOF. As announced before, we apply transfer Proposition 3.2.1. The first condition holds by assumption, the second condition holds bij Proposition 6.2.4, and the third condition holds by Proposition 6.2.6. Note that (4) holds by Proposition 6.2.6. To show that (5) holds, again we use Boolean localization. Since (5) holds for sheaves on Boolean algebras, it suffices to show that it is a geometric formula. By Proposition 6.2.5 the left adjoint of L is definable with a geometric sketch. This gives that (5) is of the form

$$\forall_f \forall_g [g \text{ is a pushout of } L(f) \Rightarrow R(g) \text{ is a weak equivalence}]$$

Since L and W are defined using geometric formulas, this is a geometric sequent. Therefore, by Boolean localization it holds in all toposes which allows us to conclude the theorem. □

7.3. Examples of Sheafifying Homotopy

Now we look at how we can apply this theorem in concrete situations. The main difficulties in applying it are finding the generating set for the cofibrations and solving the solution set condition. In many examples the cofibrations are the monomorphisms, and in this case we can easily apply the theorem. This is due the fact that in certain categories we can always find generators for the monomorphisms.

Proposition 7.3.1. *In a topos $\mathscr{E}$ the monomorphisms are the cofibrations generated by some set I.*

PROOF. Recall that toposes are locally presentable by Theorem 2.2.8.

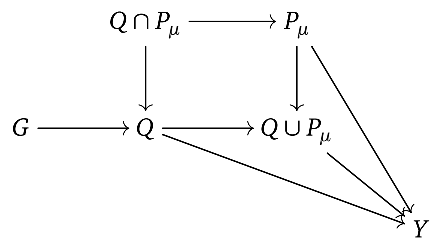
Let S be a set of objects such that every object of $\mathscr{E}$ can be written as a colimit of objects in S. Define $\mathscr{Q}$ to be the set of coequalizers of kernel pairs of epimorphisms between objects in S. So, if $e : G_1 \to G_2$ is an epimorphism between objects in S, then we can form its kernel pair by forming the pullback

$$\begin{array}{ccc} P & \xrightarrow{p} & G_1 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle e} \\ G_1 & \xrightarrow[e]{} & G_2 \end{array}$$

Then for $\mathscr{Q}$ we look at the coequalizer of p and q for all epimorphisms $e : G_1 \to G_2$ between objects in S. Now define I to be the set of subobjects of objects in Q.

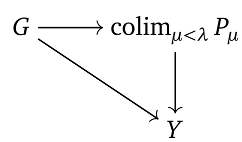
Let $f : X \to Y$ be a monomorphism. Define a set T to be the set of all arrows $G \to Y$ with $G \in S$, and let λ be the cardinality. Now we define objects P_μ for $\mu < \lambda$ using transfinite induction such that we can factor f as $X \to P_\mu \to Y$ and the arrow $X \to P_\mu$ is a monomorphism. Also, every map $X \to P_\mu$ must be a I-cellular complex on X, and we must have maps $P_\mu \to P_{\mu+1}$. We define P_0 to be X and f is the map $P_0 \to Y$. Next define $P_{\mu+1}$. Since the set T has cardinality λ, for each $\mu < \lambda$ we can find a unique $f : G \to Y$. By Proposition 5.1.8 we can factor this as $G \to Q \to Y$ with $G \to Q$ epi and

$Q \to Y$ mono. Now we can consider the diagram



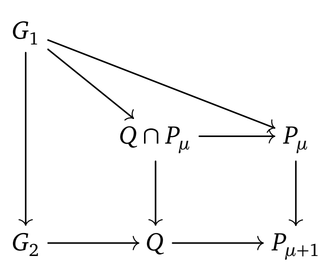
And we define $P_{\mu+1}$ as $Q \cup P_\mu$. Note that Q lies in $\mathcal{Q}$, because it is constructed as a coequalizer of a kernel pair of epimorphisms. So, $X \to P_{\mu+1}$ is a I-cellular complex, because it is a pushout of an I-cellular complex along a map in I. Lastly, for a limit ordinal ν we define P_ν to be the colimit of P_μ for $\mu < \nu$.

Our goal is to show that $P_\lambda = \operatorname{colim}_{\mu<\lambda} P_\mu$. Note that by construction we have a monomorphism $\operatorname{colim}_{\mu<\lambda} P_\mu \to Y$, and if we show that this monomorphism has a section, then we are finished. Firstly, we need to show that the following diagram commutes

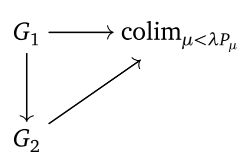


This commutes by construction. We can map G into some P_μ, and if we look at the construction of $P_{\mu+1}$, then we see it commutes. From this we can conclude that the map will indeed be a section if we can construct it.

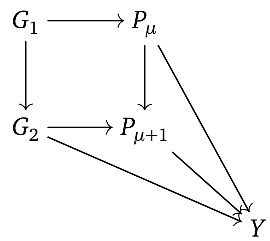
Secondly, we need to show that we have a map $Y \to \operatorname{colim}_{\mu<\lambda} P_\mu$. If we have two maps $f_1 : G_1 \to Y$ and $f_2 : G_2 \to Y$, then we get two maps $G_1 \to \operatorname{colim}_{\mu<\lambda} P_\mu$ and $G_2 \to \operatorname{colim}_{\mu<\lambda} P_\mu$. Now there are two cases, because either f_1 or f_2 was considered first in the construction. We have a bijection $\varphi : T \to \lambda$ and either $\varphi(f_1) < \varphi(f_2)$ or $\varphi(f_2) < \varphi(f_1)$. If f_1 was considered first, so if $\varphi(f_1) < \varphi(f_2)$, then we get the diagram



All triangles commute, so the diagram



commutes as well. In the other case, f_2 was considered first, so $\varphi(f_2) < \varphi(f_1)$, and the we get the diagram



Since the lower two triangles commute and the map $P_{\mu+1} \to Y$, it follows that the square commutes. □

Let us now give two example applications of Theorems 7.2.1 and 7.2.2.

Example 7.3.2. Recall from Example 2.1.4 that we have a model structure on **SSet**. The cofibrations are defined as the monomorphisms and by the previous proposition these are generated by a set in every topos. Also, we can define using a geometric statement that a map is a monomorphism. Weak equivalences between Kan complexes are the maps which induce isomorphisms on all homotopy group, and for this we need to say that it is both injective and surjective. So, if we have Kan complexes X and Y and a map $f : X \to Y$, then with this we can say using a geometric definition that f is a weak equivalence. If we have an arbitrary 0-simplex x_0 and two n-simplices x_n and x'_n whose 0-faces are x_0, then x_n and x'_n are homotopic if $f(x_n)$ and $f(x'_n)$ are homotopic. This means that f induces an injection on the homotopy groups, and similarly we can state that f induces a surjection on the homotopy groups.

For arbitrary simplicial sets we need the Ex^∞ functor. The point is that $\mathrm{Ex}^\infty(X)$ is a fibrant replacement of X and that Ex^∞ can be defined using finite limits and a colimit. So, if we have a simplicial map $f : X \to Y$, then f is a weak equivalence iff $\mathrm{Ex}^\infty(f)$ is. With a geometric definition we can say that $\mathrm{Ex}^\infty(f)$ is a weak equivalence, and thus the weak equivalences of simplicial sets are definable using geometric definitions. Now we can apply Theorem 7.2.1 to conclude this argument.

Example 7.3.3. In Section 3.2 we defined a left adjoint of the i : **SAlg** → **SSet**, and for **Sets** this could be used to transfer the model structure. This functor is also definable which we showed in Example 6.2.3. By the previous example we have a model structure on all simplicial sheaves, and therefore we can transfer this model structure using Theorem 7.2.2.

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